

SEMIGROUP ACTIONS OF EXPANDING MAPS

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ABSTRACT. We consider semigroups of Ruelle-expanding maps, parameterized by random walks on the free semigroup, with the aim of examining their complexity and exploring the relation between intrinsic properties of the semigroup action and the thermodynamic formalism of the associated skew-product. In particular, we clarify the connection between the topological entropy of the semigroup action and the growth rate of the periodic points, establish the main properties of the dynamical zeta function of the semigroup action and prove the existence of stationary probability measures.

1. INTRODUCTION

In the mid seventies the thermodynamic formalism was brought from statistical mechanics to dynamical systems by the pioneering work of Sinai, Ruelle and Bowen [27, 7, 31]. The correspondence between one-dimensional lattices and uniformly hyperbolic dynamics conveyed several notions from one setting to the other, introducing, via Markov partitions, Gibbs measures and equilibrium states into the realm of dynamical systems; see, for instance, [8]. Within non-invertible dynamics, a complete description of the thermodynamic formalism has been established for Ruelle-expanding maps [28] and for expansive maps with a specification property [30, 17]. In particular, it is known that for every potential under some regularity condition there exists a unique equilibrium state, which is a Gibbs measure and has exponential decay of correlations.

The classical strategy to prove these properties ultimately relies on the analysis of the spectral properties of the Perron-Fröbenius transfer operator, and we may extend this method to finitely generated group actions. Yet, the attempts to generalize the previous results have so far been riddled with difficulties, and a global theory is still unknown. Some success has been registered within continuous actions of finitely generated abelian groups. More precisely, the statistical mechanics of expansive \mathbb{Z}^d -actions satisfying a specification property has been studied by Ruelle in [26], after introducing a suitable notion of pressure and discussing its link with measure theoretical entropy and free energy. The crucial ingredient in this context has been the fact that continuous \mathbb{Z}^d -actions on compact spaces admit probability measures invariant under every continuous map involved in the group action. With it, Ruelle proved a variational principle for the topological pressure and built equilibrium states as the class of pressure maximizing invariant probability measures. This duality between topological and measure theoretical complexity of the dynamical

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system has been later used by Eizenberg, Kifer and Weiss [14] to establish large deviations principles for \mathbb{Z}^d -actions satisfying a specification property.

A unified approach to the thermodynamic formalism for continuous group actions in the absence of probability measures invariant under all elements of the group is still incomplete. Although these actions are not dynamical systems, a few definitions of topological pressure have been proposed, although most of them unrelated and assuming either abelianity, amenability or some growth rate of the corresponding group. Inspired by the notion of complexity presented by Bufetov in [10], in the context of skew-products, where no commutativity or conditions on the semigroup growth rate are required, the second and third named authors introduced in [24] a notion of topological pressure for a semigroup action and showed that it reflects the complex behavior of the action. In the present paper we push this analysis further, considering semigroups of Ruelle-expanding maps. We relate the notion of topological entropy of a semigroup action introduced in [24] with the growth of the set of periodic orbits, the concepts of fibered and relative entropies and the radius of convergence of a dynamical zeta function for the semigroup action. Moreover, we generalize the classical Ruelle-Perron-Fröbenius transfer operator and construct stationary measures and equilibrium states for semigroup actions of C^2 expanding maps. Meanwhile, we examine how the choice of the random walk in the semigroup unsettles the ergodic properties of the action.

2. SETTING

Let M be a compact metric space and $C^0(M)$ denote the space of all continuous observable functions $\psi : M \rightarrow \mathbb{R}$. Given a finite set of continuous maps $g_i : M \rightarrow M$, $i \in \{1, 2, \dots, p\}$, $p \geq 1$, and the finitely generated semigroup (G, \circ) with the finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_p\}$, we will write

$$G = \bigcup_{n \in \mathbb{N}_0} G_n$$

where $G_0 = \{id\}$ and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity's sake we will use $g_j g_i$ instead of the composition $g_j \circ g_i$). A semigroup can have multiple generating sets. We will assume that the generator set G_1 is minimal, meaning that no function g_j , for $j = 1, \dots, p$, can be expressed as a composition from the remaining generators.

Free semigroups. Observe that each element \underline{g} of G_n may be seen as a word that originates from the concatenation of n elements in G_1 . Clearly, different concatenations may generate the same element in G . Nevertheless, in most of the computations to be done, we shall consider different concatenations instead of the elements in G they create. One way to interpret this statement is to consider the itinerary map

$$\begin{aligned} \iota : \quad F_p &\rightarrow G \\ \underline{i} = i_n \dots i_1 &\mapsto \underline{g}_{\underline{i}} := g_{i_n} \dots g_{i_1} \end{aligned}$$

where F_p is the free semigroup with p generators, and to regard concatenations on G as images by ι of paths on F_p .

Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let G_n^* denote the space of concatenations of n elements in G_1^* . To summon each element \underline{g} of G_n^* , we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$. In G , one consider the semigroup operation of concatenation defined as usual: if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$ and $\underline{h} = h_{i_m} \dots h_{i_2} h_{i_1}$, where $n = |\underline{g}|$ and $m = |\underline{h}|$, then

$$\underline{g}\underline{h} = g_{i_n} \dots g_{i_2} g_{i_1} h_{i_m} \dots h_{i_2} h_{i_1} \in G_{m+n}^*.$$

We then say that the finitely generated semigroup G induces a *semigroup action* $S : G \times M \rightarrow M$ in M if, for any $\underline{g}, \underline{h} \in G$ and all $x \in M$, we have $S(\underline{g}\underline{h}, x) = S(\underline{g}, S(\underline{h}, x))$. The action S is said to be continuous if, for any $\underline{g} \in G$, the map $\underline{g} : M \rightarrow M$ given by $\underline{g}(x) = S(\underline{g}, x)$ is continuous. As usual, $x \in M$ is said to be a *fixed point* for $\underline{g} \in G$ if $\underline{g}(x) = x$; the set of these fixed points will be denoted by $\text{Fix}(\underline{g})$. A point $x \in M$ is said to be a *periodic point of period n* if there exists $\underline{g} \in G_n^*$ such that $\underline{g}(x) = x$. We let $\text{Per}(G_n) = \bigcup_{|\underline{g}|=n} \text{Fix}(\underline{g})$ denote the set of all periodic points of period n . Accordingly, $\text{Per}(G) = \bigcup_{n \geq 1} \text{Per}(G_n)$ will stand for the set of periodic points of the whole semigroup action. We observe that, when $G_1^* = \{f\}$, this definition coincides with the usual one of periodic points for the dynamical system f .

Ruelle-expanding maps. Let (X, d) be a compact metric space and $T : X \rightarrow X$ a continuous map. T is said to be *expansive* if there exist $\varepsilon > 0$ such that, for any $x, y \in X$,

$$\sup_{n \in \mathbb{N}} d(T^n(x), T^n(y)) < \varepsilon \quad \Rightarrow \quad x = y.$$

The map T is *locally expanding* if there exist $\lambda > 1$ and $\delta > 0$ such that, for any $x, y \in X$,

$$d(x, y) < \delta \quad \Rightarrow \quad d(T(x), T(y)) > \lambda d(x, y).$$

These two notions are bonded: every locally expanding system is expansive and an expansive map is locally expanding with respect to an adapted metric (cf. [13]).

Definition 2.1. The system (X, T) is *Ruelle-expanding* if T is locally expanding and open, that is:

- (1) There exists $c > 0$ such that, for all $x, y \in X$ with $x \neq y$, we have

$$T(x) = T(y) \quad \Rightarrow \quad d(x, y) > c.$$

- (2) There are $r > 0$ and $0 < \rho < 1$ such that, for each $x \in X$ and all $a \in T^{-1}(\{x\})$ there is a map $\varphi : B_r(x) \rightarrow X$, defined on the open ball centered at x with radius r , such that $\varphi(x) = a$, $T \circ \varphi(z) = z$ and, for all $z, w \in B_r(x)$, we have

$$d(\varphi(z), \varphi(w)) \leq \rho d(z, w).$$

In Section 4, we will recall the most relevant topological and ergodic properties of Ruelle-expanding maps. (The reader acquainted with these results may omit this section.)

3. STATEMENT OF THE MAIN RESULTS

We start with a topological description of finitely generated semigroups of uniformly expanding maps. Later, we will begin assuming that G_1 is either a finite subset of Ruelle-expanding maps acting on a compact connected metric space M or a finite subset of the space $End^2(M)$ of non-singular C^2 endomorphisms in a compact connected Riemannian manifold M . In this setting, we will show that the set of periodic points with period n for such a semigroup dynamics has a definite exponential growth rate with the period n , which is given by the topological entropy of the semigroup (see Definition 6.1) and this is equal to the logarithm of the spectral radius of a suitable choice of Ruelle-Perron-Fröbenius transfer operators $(\mathbf{L}_{n,0})_{n \geq 1}$ for the semigroup action (we refer the reader to Subsections 5 and 6 for the precise definitions of these concepts).

Theorem A. *Let G be the semigroup generated by a set $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a set of Ruelle-expanding maps on a compact connected metric space M and let $S : G \times M \rightarrow M$ be its continuous semigroup action. Then*

$$0 < h_{top}(S) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g}) \right) = sp((\mathbf{L}_{n,0})_{n \in \mathbb{N}}).$$

The second part of this work concerns the asymptotic growth of the periodic points of a semigroup action. Assume that the periodic points for a continuous mapping $f : M \rightarrow M$ are isolated. Then, following [2], the Artin-Mazur zeta function ζ_f of f is defined as

$$\zeta_f(z) = \exp \left(\sum_{n=1}^{+\infty} \frac{\# \text{Per}_n(f)}{n} z^n \right)$$

with z a complex variable and $\# \text{Per}_n(f)$ standing for the the number of periodic points of f with period n . In a similar way, given a semigroup G generated by a set $G_1 = \{g_1, \dots, g_p\}$ of maps on a Riemannian manifold M and the corresponding semigroup action $S : G \times M \rightarrow M$, we define the zeta function of G by

$$z \in \mathbb{C} \quad \mapsto \quad \zeta_S(z) = \exp \left(\sum_{n=1}^{\infty} \frac{N_n(G)}{n} z^n \right)$$

where

$$N_n(G) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g}).$$

The natural queries about this function are motivated by Subsection 4. There, we will recall that, under suitable conditions, the sequence $(N_n(G))_{n \in \mathbb{N}}$ has a definite rate of growth

$$\wp(S) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log (\max\{N_n(G), 1\})$$

and that the function ζ_S is rational, and so has a meromorphic extension to the complex plane. In general, what kind of function is ζ_S ? How is its radius of convergence related with the topological entropy of the semigroup action?

Theorem B. *Let G be the semigroup generated by a set $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a set of Ruelle-expanding maps on a compact connected metric space M and $S : G \times M \rightarrow M$ the corresponding continuous semigroup action on M . Then the dynamical zeta function ζ_S is rational and its radius of convergence is $\rho_S = e^{-\varphi(S)} = e^{-h_{top}(S)}$.*

Afterwards, we study the ergodic properties of semigroups of maps in $End^2(M)$. Although one does not expect to find an absolutely continuous common invariant probability measure, we may hope to discover some probability measure which reflects an averaged distribution of the Lebesgue measure under the action of the semigroup. We say that $R_{\underline{\theta}}$ is a *random walk on G* if $R_{\underline{\theta}} = \iota_* \theta^{\mathbb{N}}$, where θ is a probability measure on $\iota^{-1}(G_1^*) = \{1, \dots, p\}$, the set of generators of F_p . For instance, if $\theta(i) = \frac{1}{p}$ for any $i \in \{1, \dots, p\}$, then $\eta_{\underline{p}} = \theta^{\mathbb{N}}$ is the equally distributed Bernoulli probability measure on the Borel sets of the unilateral shift $\Sigma_p^+ = \{1, \dots, p\}^{\mathbb{N}}$. If, instead, $\theta(i) = a_i > 0$ for each $i \in \{1, \dots, p\}$, then $\underline{a} = (a_1, a_2, \dots, a_p)$ is a non-trivial probability vector and $\eta_{\underline{a}} = \theta^{\mathbb{N}}$ will stand for the Bernoulli probability measure $\theta^{\mathbb{N}}$ on Σ_p^+ , while $R_{\underline{a}} = \iota_*(\eta_{\underline{a}})$ will denote the corresponding random walk $R_{\underline{\theta}}$ on G . More generally, we may take a σ -invariant probability measure η on Σ_p^+ and consider the associated random walk $R_{\eta} = \iota_*(\eta)$. Given a σ -invariant probability measure η , a probability measure ν on M is said to be *R_{η} -stationary* if

$$\nu = \int \underline{g}_* \nu \, dR_{\eta}(\underline{g})$$

that is, for every continuous observable $\psi : M \rightarrow \mathbb{R}$, we have

$$\int \psi \, d\nu = \int \left(\int \psi \circ \underline{g} \, d\nu \right) dR_{\eta}(\underline{g}). \quad (1)$$

We will show that every semigroup action of C^2 expanding maps admits a unique stationary measure which is absolutely continuous with respect to the Lebesgue measure and whose density can be obtained via the iterations of a Ruelle-Perron-Fröbenius operator. Finally, we discuss a thermodynamic formalism and find an adequate definition of measure of maximal entropy for a semigroup action with respect to a fixed random walk.

Theorem C. *Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$ of C^2 expanding maps on a compact connected Riemannian manifold M . Consider the corresponding continuous semigroup action $S : G \times M \rightarrow M$ and the random walk $R_{\underline{p}} = \iota_*(\eta_{\underline{p}})$. Then, the semigroup action has:*

- an absolutely continuous stationary probability measure ν_1 ;
- a probability measure of maximal entropy ν_0 .

Moreover, the maximal entropy measure can be computed as the weak-limit of an averaged distribution of either preimages or periodic points.*

OVERVIEW

In Section 4 we recall the main properties of Ruelle-expanding maps, both as a guide to the reader and for reference in the remaining of the text. The reader that is acquainted

with the classical theory of Ruelle expanding maps may choose to skip this section in a first reading of the text.

A sequence of Ruelle-Perron-Frobenius transfer operators suitable for the analysis of semigroup actions is defined in Section 5. In the case of random walks on the free semigroup, this sequence of operators coincides with the iterates of a Ruelle-Perron-Frobenius transfer that averages the transfer operators of the dynamical systems that generate the semigroup. Moreover, these operators are in a strong relation with the classical transfer operator for a locally constant skew-product dynamics. To complement the classical approach of random dynamical systems, we are mainly interested in determining intrinsic objects for the dynamics of the semigroup action. For that reason, we will, in particular, discuss the dependence of the invariant measures for skew-product dynamics on the probability measures within the underlying shift that describes the random walk on the free semigroup.

In Sections 6 and 7 we justify the notion of topological entropy for semigroup actions. Indeed, using the relation with the skew product dynamics, we prove not only that the entropy can be computed by the growth rate of the mean number of periodic orbits but also that it arises naturally as the radius of convergence of the zeta function (Theorems A and B).

Motivated by the strong connections found between the skew-product dynamics and the semigroup actions, in Sections 8 and 9 we focus on building a bridge between the several concepts of the thermodynamic formalism for skew-products and the intrinsic objects for semigroup actions. In particular, we compare the notions of classical entropy, fibered entropy, relative entropy, and quenched and annealed equilibrium states for symmetric and non-symmetric random walks. For instance, we prove that the entropy of the semigroup coincides with the fibered entropy of Ledrappier and Walters [18] if and only if all maps have the same degree (Proposition 8.3); and that the latter coincides with a quenched pressure in random dynamical systems (Proposition 8.10). The entropy of the semigroup is showed to coincide with an annealed pressure in random dynamical systems (Corollary 8.9) and also to be equal to the classical topological pressure of a suitable potential for the skew-product dynamics (cf. (28)). Some results on stationary measures are recalled as well while constructing absolutely continuous stationary measures in Theorem C.

In Section 10 we select measures for the semigroup action using some variational insight. Using both connections between the semigroup action, the classical thermodynamic formalism and the annealed pressure function, we provide an intrinsic construction of measures (not necessarily stationary) on the ambient space which arise as marginals from equilibrium states. The two different approaches allow us to conclude that such maximal entropy measures can be computed using either an averaged equidistribution of periodic points or of preimages, which ends the proof of Theorem C.

4. MAIN PROPERTIES OF RUELLE-EXPANDING MAPS

Let (X, d) be a compact metric space and $T : X \rightarrow X$ a Ruelle-expanding map.

Expansivity. Any ε verifying $\varepsilon < \min \{r, \frac{c}{1+\rho}\}$ is a constant of expansivity of T . This means that

$$d(T^n(x), T^n(y)) \leq \varepsilon \quad \forall n \geq 0 \quad \Rightarrow \quad x = y.$$

This property ensures that, for any $n \in \mathbb{N}$, the periodic points with period n are isolated and therefore in finite number. Yet, in this context, we also have

$$X = \bigcup_{n \geq 0} T^{-n}(\overline{\text{Per}(T)})$$

where $\text{Per}(T)$ denotes the set of periodic points of T (details in [12]).

Pre-images. The cardinal of the pre-images by T is uniformly bounded (that is, there is $p \in \mathbb{N}$ such that $\text{card}(T^{-1}(\{x\})) \leq p$, for all $x \in X$) and locally constant. Moreover, if $\delta = \min\{r, \frac{c}{2\rho}\}$, and $x, y \in X$ are such that $d(x, y) < \delta$, then, for any positive integer n , we may write

$$T^{-n}(x) = \{x_1, x_2, \dots, x_{k_n}\} \quad \text{and} \quad T^{-n}(y) = \{y_1, y_2, \dots, y_{k_n}\}$$

satisfying $d(T^j(x_i), T^j(y_i)) \leq \rho^{n-j} d(x, y)$ for all $0 \leq j \leq n$ and $1 \leq i \leq k_n$. (See [12] for details.)

Contractive branches. Let $S \subseteq X$. Given $n \in \mathbb{N}$, we say that $\phi : S \rightarrow X$ is a *contractive branch* of T^{-n} if

- (1) $(T^n \circ \phi)(x) = x \quad \forall x \in S$
- (2) $d((T^j \circ \phi)(x), (T^j \circ \phi)(y)) \leq \rho^{n-j} d(x, y) \quad \forall x, y \in S \quad \forall j \in \{0, 1, \dots, n\}$.

It is straightforward (cf. [12]) to conclude that, given $x \in X$, $n \in \mathbb{N}$ and $a \in T^{-n}(x)$, there is a contractive branch $\phi : B_r(x) \rightarrow X$ of T^{-n} such that $\phi(x) = a$. Moreover, if ε is a constant of expansivity, $\varsigma \leq \varepsilon$ and $B_{n, \varsigma, a} = \{z \in X : d(T^j(z), T^j(a)) < \varsigma \quad \forall 0 \leq j \leq n\}$, then $B_{n, \varsigma, a} = \phi(B_\varsigma(x))$.

Spectral decomposition of the dynamics. In [28], it was proved that there exists a (unique) finite family $(\Lambda_i^{(m)})_{i \in \{1, \dots, n_m\}; m \in \{1, \dots, M\}}$ of compact disjoint subsets, called basic components, such that

- (C1) $T(\Lambda_i^{(m)}) = \Lambda_{i+1}^{(m)}$ for all $i \in \{1, \dots, n_m - 1\}$ and $m \in \{1, \dots, M\}$.
- (C2) $T(\Lambda_{n_m}^{(m)}) = \Lambda_1^{(m)}$ for all $m \in \{1, \dots, M\}$.
- (C3) $\bigcup_{i, m} \Lambda_i^{(m)} = \overline{\text{Per}(T)}$.
- (C4) $T|_{\Lambda_i^{(m)}}$ is Ruelle-expanding.
- (C5) For any open non-empty subset V of $\Lambda_i^{(m)}$ there is $N \in \mathbb{N}$ such that $(T^{n_m})^N(V) = \Lambda_i^{(m)}$.

Notice that, if X is connected, then $X = \overline{\text{Per}(T)}$ and X reduces to one basic component (where, by (C5), T is topologically mixing). Following [20], we say that a map $T : X \rightarrow X$ is covering if for any non-empty open set V there exists an iterate $N \in \mathbb{N}$ such that $T^N(V) = X$. From (C5), every Ruelle-expanding topologically mixing map is covering.

Dynamical zeta function and entropy. The zeta function of T is a formal series that encodes the information regarding the number of periodic points of T , given by

$$z \in \mathbb{C} \mapsto \zeta_T(z) = \exp \left(\sum_{n=1}^{\infty} \frac{N_n(T)}{n} z^n \right)$$

where $N_n(T)$ is the number of periodic points with period n , for all $n \in \mathbb{N}$ (cf. [22] and references therein). In our context, ζ_T is a rational function, so it has a meromorphic continuation to the whole complex plane. The poles, zeros and residues of this extension provide additional topological invariants for T and an insight into its orbit structure. In particular, the topological entropy of T is equal to $-\log(\varrho)$, where ϱ is the radius of convergence of ζ_T . Additionally, if T is topologically mixing, then

$$h_{\text{top}}(T) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log N_n(T)$$

a limit equal to zero if and only if X is finite. The reader may find more details in [12].

Thermodynamic formalism. Given a positive constant $\theta \in]0, 1[$, consider the space of Hölder-continuous maps with exponent θ

$$\mathfrak{F}_{\theta}^+ = \{w : X \rightarrow \mathbb{R} : \exists C_w > 0 : |w(x) - w(y)| \leq C_w d(x, y)^{\theta} \quad \forall x, y \in X\}$$

with the norm

$$\|w\|_{\theta} = \|w\|_{\infty} + \sup_{x \neq y, d(x, y) < \delta} \frac{|w(x) - w(y)|}{d(x, y)^{\theta}}$$

where $\delta = \min\{r, \frac{c}{2\rho}\}$. The space of real continuous functions defined in X , with the uniform norm $\|\cdot\|_0$, will be denoted by $C^0(X)$. For a real valued function $\varphi : X \rightarrow \mathbb{R}$, the associated *Ruelle operator* \mathfrak{L}_{φ} acts on a map $\psi : X \rightarrow \mathbb{R}$ as

$$x \in X \mapsto \mathfrak{L}_{\varphi}(\psi)(x) = \sum_{y \in T^{-1}(x)} \psi(y) e^{\varphi(y)}.$$

This operator is well defined (see the properties of the pre-images of T), linear and positive. By induction, we may verify that, for any ψ and x ,

$$\mathfrak{L}_{\varphi}^n(\psi)(x) = \sum_{y \in T^{-n}(x)} \psi(y) e^{S_n \varphi(y)}$$

where $S_n \varphi(y) = \varphi(y) + \dots + \varphi(T^{n-1}y)$. If $\varphi \in C^0(X)$, then $\mathfrak{L}_{\varphi}(\psi) \in C^0(X)$ and the operator \mathfrak{L}_{φ} restricted to $C^0(X)$ is continuous with respect to the C^0 -norm. Similarly, if $\varphi, \psi \in \mathfrak{F}_{\theta}^+$, then $\mathfrak{L}_{\varphi}(\psi) \in \mathfrak{F}_{\theta}^+$. The most significant ergodic properties of Ruelle-expanding maps yielded by the associated Ruelle operator are described by the following theorem.

Theorem 4.1. [28] *Assume that $T : X \rightarrow X$ is a topologically mixing Ruelle-expanding map and $\varphi \in \mathfrak{F}_{\theta}^+$. Then:*

- (1) *There is a simple maximal positive eigenvalue λ_{φ} of $\mathfrak{L}_{\varphi} : C^0(X) \rightarrow C^0(X)$ with a corresponding strictly positive eigenfunction $H \in \mathfrak{F}_{\theta}^+$.*
- (2) *H is the unique positive eigenfunction of \mathfrak{L}_{φ} , up to scalar multiplication.*

- (3) The remainder of the spectrum of $\mathfrak{L}_\varphi : \mathfrak{F}_\theta^+ \rightarrow \mathfrak{F}_\theta^+$ is contained in a disk centered at $(0, 0)$ with radius $R < \lambda_\varphi$.
- (4) There is a unique probability ν on the Borel subsets of X such that $\mathfrak{L}_\varphi^* \nu = \lambda_\varphi \nu$ and $\int H d\nu = 1$.
- (5) For any $\psi \in C^0(X)$, $\lim_{n \rightarrow +\infty} \|\lambda_\varphi^{-n} \mathfrak{L}_\varphi^n(\psi) - H \int \psi d\nu\|_0 = 0$.
- (6) For any $\psi \in \mathfrak{F}_\theta^+$, $\sup_{n \in \mathbb{N}} \|\lambda_\varphi^{-n} \mathfrak{L}_\varphi^n(\psi)\|_\theta < \infty$.
- (7) The probability $\mu = H\nu$ is T -invariant, exact, positive on non-empty open sets and satisfies a variational principle: it is the unique probability such that

$$\log(\lambda_\varphi) = h_\mu(T) + \int \varphi d\mu = \max_{u \in \mathfrak{M}(X, T)} \left(h_u(T) + \int \varphi du \right)$$

where $\mathfrak{M}(X, T)$ denotes the (compact, convex) space of T -invariant probability measures on the σ -algebra of the Borel subsets of X .

Remark 4.2. In the particular case of $T : X \rightarrow X$ being a C^2 expanding map of a compact connected Riemannian manifold X endowed with a Lebesgue probability measure (Leb for short), we may consider $\varphi : X \rightarrow \mathbb{R}$ given by $\varphi(x) = -\log |\det DT_x|$, which belongs to \mathfrak{F}_θ^+ . Then there is a unique probability measure μ on the Borel sets of X which is invariant under T and absolutely continuous with respect to Lebesgue. Moreover, the density of μ with respect to Lebesgue is precisely given by H , so it is Hölder and strictly positive. Additionally, μ is exact; $h_\mu(T) = \int \log |\det DT_x| d\mu$; $h_u(T) < \int \log |\det DT_x| du$, for every T -invariant probability measure $u \neq \mu$; and, for any Borel set A , we have $\lim_{n \rightarrow +\infty} \text{Leb}(T^{-n}(A)) = \mu(A)$.

If, instead, we consider $\varphi \equiv 0$ and X is connected, then we may deduce relevant information concerning the asymptotic distribution of the pre-images of each point by the expanding map $T : X \rightarrow X$. Given $x \in X$ and $n \in \mathbb{N}$, let $\nu_n(x)$ be the probability measure on the Borel subsets of X defined as

$$\nu_n(x) = \frac{1}{\deg(T)^n} \sum_{T^n(y)=x} \delta_y \quad (2)$$

where δ_y is the Dirac measure supported on $\{y\}$ and $\deg(T)$ is the degree of T , that is, $\deg(T) = \#T^{-1}(\{a\})$, a number independent of $a \in X$. Then there exists a unique probability measure ν on the Borel sets of X such that, for every $x \in X$,

$$\nu = \lim_{n \rightarrow +\infty} \nu_n(x)$$

in the weak* topology. Such a ν is invariant under T (that is, $H \equiv 1$), exact, positive on non-empty open sets, its entropy is equal to $h_\nu(T) = \log(\deg(T))$ (that is, $\lambda_\nu = \deg(T)$) and it maximizes entropy (that is, $h_\nu(T) > h_v(T)$ for every T -invariant probability measure $v \neq \nu$). Moreover, $\nu = \mu$ if and only if, for all $x \in X$ such that $T^n(x) = x$, we have $|\det D_x(T^n)| = \deg(T)^n$. We refer the reader to [8] for more information.

Gibbs measures. Each equilibrium state μ obtained in Ruelle's Theorem has a positive Jacobian, namely $J_\mu f = e^{-\varphi}$, so it is positive on open non-empty sets. Besides, it is a Gibbs measure, that is, given a finite partition $\mathcal{P} = \{P_1, \dots, P_\ell\}$ of X with diameter less

than r , there exists a constant $C > 1$ such that, for each positive integer $m \geq 0$, any $1 \leq i \leq \ell$, every contractive branch $\phi : P_i \rightarrow X$ of T^m and all $x \in \phi(P_i)$, we have

$$C^{-1} \leq \frac{\mu(\phi(P_i))}{e^{-m \log(\lambda_\varphi) + S_m \varphi(x)}} \leq C.$$

Examples. One-sided Markov subshifts of finite type, determined by aperiodic square matrices with entries in $\{0, 1\}$, are Ruelle-expanding map (with $r = 1$ and $c = \rho = 1/2$) and topologically mixing [37]. If M is a compact Riemannian manifold without boundary and $T : M \rightarrow M$ a C^1 map, the dynamical system T is said to be C^1 -expanding if

$$\exists \lambda > 1 : \forall x \in M \forall v \in T_x M \quad \|D_x T(v)\| \geq \lambda \|v\|.$$

It is easy to prove that, in the C^1 context, T is C^1 -expanding if and only if it is Ruelle-expanding. For instance, $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, $T(z) = z^m$, is C^1 -expanding for all positive integer $m > 1$ (in this case $\lambda = m$). Moreover, if $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a C^2 map with degree bigger than one such that $DT(z) \neq 0$ for all $z \in \mathbb{S}^1$ and all the periodic points of T are hyperbolic (a generic property), then the restriction of T to the complement of the union of the basins of the sinks is Ruelle-expanding. More generally, if $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map whose eigenvalues have absolute value bigger than one and such that $L(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$, then L induces a Ruelle-expanding map on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. Conversely, any C^1 -expanding map in the n -dimensional flat torus is topologically conjugate to one obtained by this process; see [32].

5. RUELLE-PERRON-FRÖBENIUS OPERATORS

In this section we shall present a proposal for the notion of Ruelle-Perron-Fröbenius transfer operator to be assigned to a semigroup action which is a natural extension of the concept of transfer operator for an individual dynamical system (cf. Remark 5.3). The operator we will introduce depends on the chosen set G_1 of generators of G and on the selected random walk R_η on G ; we will come back to this subject on Subsection 8.2.

Let G be a semigroup generated by a finite subset G_1 of Ruelle-expanding maps acting on a compact connected metric space M . Consider the corresponding continuous semigroup action $S : G \times M \rightarrow M$. Given a continuous observable $\varphi : M \rightarrow \mathbb{R}$, let $\mathfrak{L}_{\underline{g}, \varphi} : C^0(M) \rightarrow C^0(M)$ denote the usual Ruelle-Perron-Fröbenius operator associated to the dynamical system \underline{g} and the observable φ :

$$\mathfrak{L}_{\underline{g}, \varphi}(\psi)(x) = \sum_{\underline{g}(y)=x} e^{\varphi(y)} \psi(y). \quad (3)$$

Therefore, for each $k \in \mathbb{N}$,

$$\mathfrak{L}_{\underline{g}, \varphi}^k(\psi)(x) = \sum_{\underline{g}^k(y)=x} e^{S_k \varphi(y)} \psi(y),$$

where $S_k \varphi(x) = \sum_{\ell=0}^{k-1} \varphi(\underline{g}^\ell(y))$ and $\underline{g}^\ell := (g_{i_n} \dots g_{i_1})^\ell$ for every $0 \leq \ell \leq k-1$.

Alternatively, for any $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, the operator $\mathcal{L}_{\underline{g}, \varphi} : C^0(M) \rightarrow C^0(M)$ is defined as

$$\mathcal{L}_{\underline{g}, \varphi}(\psi)(x) = \mathfrak{L}_{\underline{g}, \varphi_{\underline{g}}}(\psi)(x) = \sum_{\underline{g}(y)=x} e^{\varphi_{\underline{g}}(y)} \psi(y), \quad (4)$$

where $\varphi_{\underline{g}}(y) = \sum_{m=0}^{n-1} \varphi(\underline{g}_m(y))$, $g_0 = id$ and $\underline{g}_m = g_{i_m} \dots g_{i_1}$ for every $1 \leq m \leq n-1$. Observe that, if φ is continuous (respectively, Hölder) and the elements of G_1 are C^2 expanding maps, then $\varphi_{\underline{g}}$ is continuous (respectively, Hölder) as well. Moreover, for each $k \in \mathbb{N}$,

$$\mathcal{L}_{\underline{g}, \varphi}^k(\psi)(x) = \sum_{\underline{g}^k(y)=x} e^{S_k \varphi_{\underline{g}}(y)} \psi(y)$$

where, as usual, for every $\phi : M \rightarrow \mathbb{R}$, we write $S_k \phi(x) = \sum_{\ell=0}^{k-1} \phi(\underline{g}^\ell(y))$ and $\underline{g}^\ell = (g_{i_n} \dots g_{i_1})^\ell$ for every $0 \leq \ell \leq k-1$. Roughly speaking, the transfer operator $\mathcal{L}_{\underline{g}, \varphi}$ gathers the information of the pathwise transfer operators while we evaluate the observable along the n th iteration of the semigroup dynamics, as specified by (4). It is not hard to check that $\mathcal{L}_{\underline{g}, \varphi} = \mathfrak{L}_{g_{i_n}, \varphi} \circ \mathfrak{L}_{g_{i_{n-1}}, \varphi} \dots \circ \mathfrak{L}_{g_{i_1}, \varphi}$ for any $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$ and every positive integer n .

Remark 5.1. The operator $\mathcal{L}_{\underline{g}, \varphi}$ depends on the word that expresses \underline{g} as a combination of elements of the generator G_1 because it is based on the function $\varphi_{\underline{g}}(y) = \sum_{m=0}^{n-1} \varphi(\underline{g}_m(y))$ which may change if the order of the concatenation is altered. Thus, in this definition, we are distinguishing different concatenations even if they correspond to the same endomorphism of M .

A sequence of transfer operators. One may also consider the non-stationary dynamical system whose complexity is indexed by the ‘time n ’, corresponding to the ‘ball of radius n ’ in the semigroup. Such viewpoint has turned to be very fruitful in the description of the topological entropy and the complexity of group and semigroup actions [24], and motivates the definition of the following weighted mean sequence of transfer operators.

Definition 5.2. Given a continuous potential $\varphi : M \rightarrow \mathbb{R}$ and a continuous finitely generated semigroup action $S : G \times M \rightarrow M$, the *Ruelle-Perron-Fröbenius sequence* of G with respect to φ is the sequence $(\mathbf{L}_{n, \varphi})_{n \geq 1}$ of bounded linear operators acting on $C^0(M)$ and given, for every $n \geq 1$, by

$$\mathbf{L}_{n, \varphi} = \frac{1}{p^n} \sum_{|\underline{g}|=n} \mathcal{L}_{\underline{g}, \varphi} = \frac{1}{p^n} \sum_{|\underline{g}|=n} \mathfrak{L}_{\underline{g}, \varphi_{\underline{g}}}.$$

We observe that, for each potential φ , this sequence is obtained by averaging the usual Ruelle-Perron-Fröbenius transfer operators associated to $\varphi_{\underline{g}}$ of each dynamics \underline{g} in G_n^* . Furthermore, notice that the operator $\mathcal{L}_{\underline{g}, \varphi}$ depends on the order of the concatenation of the generators that build \underline{g} . So, we are averaging not on the maps in G_n but on the different words that express them in terms of the generators.

Example 5.3. If G is generated by $G_1 = \{id, f\}$, then $G_n^* = \{f^n\}$ for $n \geq 1$ and $\mathbf{L}_{n,\varphi} = \mathcal{L}_{f^n,\varphi} = \mathfrak{L}_{f^n,\varphi} = \mathfrak{L}_{f,\varphi}^n$. In particular, $\mathbf{L}_{1,\varphi} = \mathcal{L}_{f,\varphi} = \mathfrak{L}_{f,\varphi}$ and these three maps coincide with the usual Ruelle-Perron-Fröbenius operator for f . Therefore, they are all natural extensions of the notion of transfer operator for an individual dynamical system. Moreover, if $\varphi \equiv 0$, $\mathcal{L}_{\underline{g},0}(\psi)(x) = \sum_{\underline{g}(y)=x} \psi(y)$ and corresponds to the Ruelle-Perron-Fröbenius transfer operator $\mathfrak{L}_{\underline{g},0}$ associated to the topological entropy of $\underline{g} : M \rightarrow M$; see Subsection 4. For the same observable $\varphi \equiv 0$, the sequence of transfer operators $\mathbf{L}_{n,0}$ is given by

$$\mathbf{L}_{n,0}(\psi)(x) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \sum_{\underline{g}(y)=x} \psi(y)$$

for every $\psi \in C^0(M)$ and $x \in M$.

Averaged and fibered transfer operators for the semigroup action. Given a finitely generated continuous semigroup action $G \times M \rightarrow M$, a shift-invariant probability measure η on Σ_p^+ and a continuous potential $\varphi : M \rightarrow \mathbb{R}$, take the non-stationary sequence $(\mathbf{L}_{n,\eta,\varphi})_{n \geq 1}$ given by

$$n \geq 1 \mapsto \mathbf{L}_{n,\eta,\varphi} = \int_{\Sigma_p^+} \mathcal{L}_{g_{i_n},\varphi} \cdots \mathcal{L}_{g_{i_2},\varphi} \mathcal{L}_{g_{i_1},\varphi}(\mathbf{1}) d\eta([i_1, \dots, i_n]).$$

In the case $\eta = \eta_{\underline{p}}$, which describes the symmetric random walk, the family $(\mathbf{L}_{n,\eta_{\underline{p}},\varphi})_{n \geq 1}$ coincides with the family of transfer operators introduced in Definition 5.2. In the case of a random walk $\eta = \eta_{\underline{a}}$ associated to a non-trivial probability vector \underline{a} , the a priori non-stationary sequence becomes stationary. Indeed, given $\underline{\varphi} = (\varphi_1, \dots, \varphi_p) \in C^0(M)^p$ and a non-trivial probability vector \underline{a} , the transfer operator $\tilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}$ acting on $C^0(M)$ is precisely

$$\phi \in C^0(M) \quad \mapsto \quad \tilde{\mathbf{L}}_{\underline{a},\underline{\varphi}}\phi(x) = \sum_{i=1}^p a_i \sum_{g_i(y)=x} e^{\varphi_i(y)} \phi(y). \quad (5)$$

Averaged and fibered transfer operators for the skew-product. The semigroup actions is naturally associated to a skew-product dynamics defined by

$$\begin{aligned} \mathcal{F}_G : \Sigma_p^+ \times M &\rightarrow \Sigma_p^+ \times M \\ (\omega, x) &\mapsto (\sigma(\omega), g_{\omega_1}(x)) \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots)$. Given a non-trivial probability vector $\underline{a} = (a_1, a_2, \dots, a_p)$, where $a_k > 0$ for all $k = 1, \dots, p$ and $\sum_{k=1}^p a_k = 1$, consider the Bernoulli probability measure $\eta_{\underline{a}} = \underline{a}^{\mathbb{N}}$ on Σ_p^+ . Inspired by the work [4] on random expanding maps, we assign to any $\underline{\varphi} = (\varphi_1, \dots, \varphi_p) \in C^0(\Sigma_p^+ \times M)^p$ the integrated transfer operators

$$\hat{\mathbf{L}}_{\underline{a},\underline{\varphi}} : C^0(\Sigma_p^+ \times M) \rightarrow C^0(\Sigma_p^+ \times M)$$

defined by

$$\hat{\mathbf{L}}_{\underline{a},\underline{\varphi}}\psi(\omega, x) = \sum_{i=1}^p a_i \mathcal{L}_{g_i,\varphi_i}\psi(i\omega, x) = \sum_{i=1}^p a_i \sum_{g_i(y)=x} e^{\varphi_i(i\omega, y)} \psi(i\omega, y) \quad (6)$$

for every $\psi \in C^0(\Sigma_p^+ \times M)$, where $i\omega$ stands for the sequence $(i, \omega_1, \omega_2, \dots)$. Observe that, in the particular case of $\underline{\varphi} = (0, \dots, 0)$, one gets $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}} 1 = \int \deg(g_i) d\underline{a}(i)$. Moreover, $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^n = \mathbf{L}_{n, \eta_{\underline{a}, \underline{\varphi}}}$ for all $n \geq 1$. Furthermore, any $\varphi \in C^0(M)$ induces a sequence $\underline{\varphi}(\omega, x) = (\varphi(x), \dots, \varphi(x)) \in C^0(\Sigma_p^+ \times M)^p$; and, given $\psi \in C^0(\Sigma_p^+ \times M)$ that does not depend on ω , if we consider $\phi : M \rightarrow \mathbb{R}$ defined by $\phi(x) = \psi(1, x)$, then we have

$$\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}} \psi = \tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}} \phi. \quad (7)$$

As $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ and $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ are positive operators, their spectral radius are equal to the exponential growth rates of $\|\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^n 1_{\Sigma_p^+ \times M}\|_0$ and $\|\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^n 1_M\|_0$, respectively. Thus, $sp(\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}) = sp(\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}})$.

Remark 5.4. A similar link between transfer operators on the phase space and on the skew-product dynamics has been considered previously in [34]. However, in this reference the operators are built averaging *normalized* transfer operators for individual dynamics.

6. TOPOLOGICAL ENTROPY OF THE SEMIGROUP ACTION

This section is devoted to the proof of Theorem A, which relates the spectral radius of the Ruelle-Perron-Fröbenius sequence of G (Definition 5.2) with the exponential growth rate of periodic points and the topological entropy. First, let us recall the concept of separated points and topological entropy of a semigroup action adopted in [10, 24]. Given $\varepsilon > 0$ and $\underline{g} := g_{i_n} \dots g_{i_2} g_{i_1} \in G_n$, the *dynamical ball* $B(x, \underline{g}, \varepsilon)$ is the set

$$B(x, \underline{g}, \varepsilon) := \left\{ y \in X : d(\underline{g}_j(y), \underline{g}_j(x)) \leq \varepsilon, \text{ for every } 0 \leq j \leq n \right\}$$

where, as before, for every $1 \leq j \leq n-1$ we denote by \underline{g}_j the concatenation $g_{i_j} \dots g_{i_2} g_{i_1} \in G_j$, and $\underline{g}_0 = id$. We also assign a dynamical metric $d_{\underline{g}}$ to M by setting

$$d_{\underline{g}}(x, y) := \max_{0 \leq j \leq n} d(\underline{g}_j(x), \underline{g}_j(y)). \quad (8)$$

It is important to notice that both the dynamical ball and the metric depend on the underlying concatenation of generators $g_{i_n} \dots g_{i_1}$ and not on the group element \underline{g} , since the latter may have distinct representations.

Given $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, we say that a set $K \subset M$ is $(\underline{g}, n, \varepsilon)$ -*separated* if $d_{\underline{g}}(x, y) > \varepsilon$ for any distinct $x, y \in K$. The maximal cardinality of a $(\underline{g}, \varepsilon, n)$ -separated set on M will be denoted by $s(\underline{g}, n, \varepsilon)$. The topological entropy of a semigroup action estimates the growth rate in n of the number of orbits of length n up to some small error ε .

Definition 6.1. The *topological entropy* of the semigroup action $S : G \times M \rightarrow M$ is the limit

$$h_{\text{top}}(S) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right). \quad (9)$$

Remark 6.2. This notion of topological entropy should be compared with the one proposed by Ghys, Langevin and Walczak, where the authors compute the asymptotic exponential growth rate of points that are separated by some group element (see [15] for the precise definition). This corresponds to the largest exponential growth rate, while the definition we adopt here observes the growth rates of separated points averaged along semigroup elements.

In the context of Ruelle-expanding dynamics this notion is connected to the largest exponential growth rate of periodic points (cf. Subsection 4). This is why we also consider the following asymptotic speed.

Definition 6.3. The *periodic entropy* of the semigroup action $S : G \times M \rightarrow M$ is the limit

$$\wp(S) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log (\max\{N_n(G), 1\}) \quad (10)$$

where

$$N_n(G) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g}).$$

Observe that in order for $\wp(S)$ to be a finite value, the set $\text{Fix}(\underline{g})$ must be finite for each $\underline{g} \in G \setminus \{id\}$, which holds for instance when \underline{g} is expansive. A map $\underline{g} \in G$ is said to be *expansive* if there exists $\varepsilon_{\underline{g}} > 0$ such that, whenever $x, y \in M$ and $x \neq y$, then

$$\max \{d(\underline{g}^\ell(x), \underline{g}^\ell(y)) : \ell \in \mathbb{N}_0\} \geq \varepsilon_{\underline{g}}.$$

Within Ruelle-expanding dynamics, both notions of entropy (topological or periodic) may be estimated from the knowledge of the spectral radius of the Ruelle-Perron-Fröbenius operator. The similar concept for semigroups is as follows.

Definition 6.4. Given $\varphi \in C^0(M)$, the *spectral radius* $sp((\mathbf{L}_{n,\varphi})_{n \in \mathbb{N}})$ of the Ruelle-Perron-Fröbenius sequence $(\mathbf{L}_{n,\varphi})_{n \in \mathbb{N}}$ of positive operators in $C^0(M)$ is defined as

$$\log sp((\mathbf{L}_{n,\varphi})_n) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\mathbf{L}_{n,\varphi}(\mathbf{1})\|_{C^0}.$$

Notice that, if $\varphi \equiv 0$ and $G_1 \subset \text{End}^1(M)$, then

$$\mathbf{L}_{n,0}(\mathbf{1})(x) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \mathcal{L}_{\underline{g},0}(\mathbf{1})(x) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \# \underline{g}^{-1}(x) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \deg(\underline{g}) \quad (11)$$

After [24], we know that an action of a finitely generated semigroup of C^1 -expanding maps is strongly δ -expansive, for some $\delta > 0$, a notion that we now recall.

Definition 6.5. Given $\delta > 0$, we say that a continuous semigroup action $S : G \times M \rightarrow M$ on a compact Riemannian manifold M is *δ -expansive* if, whenever $x \neq y \in M$, there exist $\kappa \in \mathbb{N}$ and $\underline{g} \in G_\kappa$ such that $d(\underline{g}(x), \underline{g}(y)) > \delta$. S is said to be *strongly δ -expansive* if, for any $\gamma > 0$, there exists $\kappa_\gamma \geq 1$ such that, for every $x \neq y \in M$ with $d(x, y) \geq \gamma$, for all $\kappa \geq \kappa_\gamma$ and any $\underline{g} \in G_\kappa^*$, we have $d_{\underline{g}}(x, y) = \max_{0 \leq j \leq n} d(\underline{g}_j(x), \underline{g}_j(y)) > \delta$.

This is a key property that eases our task of computing the topological entropy of a semigroup action. Indeed, when the action is strongly δ -expansive, the topological entropy can be computed independently of ε . More precisely,

Lemma 6.6. [24, Theorem 25] *Let G be the semigroup generated by a set $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a finite set of Ruelle-expanding maps on a compact metric space M and $S : G \times M \rightarrow M$ its continuous semigroup action. Take $0 < \varepsilon < \delta$. Then*

$$h_{top}(S) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right).$$

Additionally, it was proved in [24] that the topological entropy is a lower bound for the exponential growth rate of periodic points. For that purpose, the authors introduced the following pathwise specification property.

Definition 6.7. We say that the continuous semigroup action $S : G \times M \rightarrow M$ associated to the finitely generated semigroup G satisfies the *(strong) orbital specification property* if, for any $\delta > 0$, there exists $T(\delta) > 0$ such that, given $k \in \mathbb{N}$, for any $\underline{h}_{p_j} \in G_{p_j}^*$ with $p_j \geq T(\delta)$ for every $1 \leq j \leq k$, for each choice of k points x_1, \dots, x_k in M , for any natural numbers n_1, \dots, n_k and any semigroup element $\underline{g}_{n_j, j} = g_{i_{n_j, j}} \dots g_{i_{2, j}} g_{i_{1, j}} \in G_{n_j}$, where $j \in \{1, \dots, k\}$, there exists $x \in M$ such that $d(\underline{g}_{\ell, 1}(x), \underline{g}_{\ell, 1}(x_1)) < \delta$ for all $1 \leq \ell \leq n_1$ and $d(\underline{g}_{\ell, j} \underline{h}_{p_{j-1}} \dots \underline{g}_{n_2, 2} \underline{h}_{p_1} \underline{g}_{n_1, 1}(x), \underline{g}_{\ell, j}(x_j)) < \delta$ for all $2 \leq j \leq k$, $1 \leq \ell \leq n_j$ where $\underline{g}_{\ell, j} = g_{i_{\ell, j}} \dots g_{i_{1, j}}$. We say that the semigroup action satisfies the *periodic orbital specification property* if the point x can be chosen periodic.

Theorem 6.8. [24, Theorem 28] *Let G be the semigroup generated by $G_1 = \{Id, g_1, \dots, g_p\}$, where G_1^* is a finite set of Ruelle-expanding maps on a compact connected metric space M and $S : G \times M \rightarrow M$ its continuous semigroup action. Then G satisfies the periodic orbital specification property and*

$$0 < h_{top}(S) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g}) \right).$$

We will show that the equality holds and that the previous lim sup is indeed a limit.

6.1. Proof of Theorem A. Let G be the semigroup generated by $G_1 = \{Id, g_1, \dots, g_p\}$, with G_1^* a finite set of Ruelle-expanding maps on a compact connected metric space M . From [24, Theorem 28], we already know that $h_{top}(S) \leq \wp(S)$. We are left to show the opposite inequality. First we will prove that the class of Ruelle-expanding maps is closed under concatenation, and so forms a semigroup.

Lemma 6.9. *If each map in the finite set G_1^* is Ruelle-expanding, then \underline{g} is Ruelle-expanding for any $\underline{g} \in G - \{Id\}$. Moreover, there exists $\delta > 0$ such that the semigroup action $S : G \times M \rightarrow M$ is strongly δ -expansive.*

Proof. Assume that g_1 and g_2 are Ruelle-expanding maps. We claim that the composition $g_2 g_1$ is a Ruelle-expanding map. Let $c_1 > 0$ and $c_2 > 0$ be such that, for all $x, y \in M$ with $x \neq y$

$$g_1(x) = g_1(y) \Rightarrow d(x, y) > c_1 \quad \text{and} \quad g_2(x) = g_2(y) \Rightarrow d(x, y) > c_2.$$

Consider the composition $g_2 g_1$ and $x, y \in X$ with $x \neq y$ and assume that $g_2 g_1(x) = g_2 g_1(y)$. Then, either $g_1(x) = g_1(y)$, in which case $d(x, y) > c_1$; or $g_1(x) \neq g_1(y)$, in which case one has $d(g_1(x), g_1(y)) > c_2$. In the latter case, observe that since g_1 is uniformly continuous, there is $\delta_1 > 0$ such that $d(g_1(a), g_1(b)) \leq c_2$ whenever $d(a, b) \leq \delta_1$. Thus, from $d(g_1(x), g_1(y)) > c_2$ we conclude that $d(x, y) > \delta_1$. Therefore, if $c_{12} = \min\{c_1, \delta_1\} > 0$, then

$$\forall x, y \in M : x \neq y, g_2 g_1(x) = g_2 g_1(y) \Rightarrow d(x, y) > c_{12}.$$

Now, let $r_1, r_2, \rho_1 < 1$ and $\rho_2 < 1$ be positive constants such that, for each $x \in M$, for every $a \in g_2^{-1}(\{x\})$ and every $b \in g_1^{-1}(\{a\})$, there exists a map $\phi_2 : B_{r_2}(x) \rightarrow M$, defined on the open ball centered at x with radius r_2 such that $\phi_2(x) = a$, $g_2 \circ \phi_2 = id$ and

$$d(\phi_2(z), \phi_2(w)) \leq \rho_2 d(z, w) \quad \forall z, w \in B_{r_2}(x),$$

so $\phi_2(B_{r_2}(x)) \subset B_{\rho_2 r_2}(a)$; and there is another map $\phi_1 : B_{r_1}(a) \rightarrow M$, defined on the open ball centered at a with radius r_1 , satisfying $\phi_1(a) = b$, $g_1 \circ \phi_1 = id$ and

$$d(\phi_1(z), \phi_1(w)) \leq \rho_1 d(z, w) \quad \forall z, w \in B_{r_1}(a).$$

The uniform contraction rate ρ_2 of ϕ_2 and the uniform size of the balls associated to the contractive branches of g_1 allow us to find $0 < r_{12} \leq r_2$ such that, for any $x \in M$ and every $a \in g_2^{-1}(\{x\})$, we have $\phi_2(B_{r_{12}}(x)) \subset B_{r_1}(a)$. For instance, we may take $r_{12} = \min\{r_1, r_2\}$. Therefore, the map $\phi_{12} : B_{r_{12}}(x) \rightarrow M$ given by $\phi_{12} \equiv \phi_1 \phi_2$ is well defined and is a contractive branch for $g_2 g_1$, since it has the following properties:

- (1) $\phi_{12}(x) = \phi_1 \phi_2(x) = \phi_1(a) = b$;
- (2) $(g_2 g_1) \circ \phi_{12} = (g_2 g_1) \circ (\phi_1 \phi_2) = g_2 (g_1 \circ \phi_1) \phi_2 = id$;
- (3) $d(\phi_{12}(z), \phi_{12}(w)) \leq \rho_1 d(\phi_2(z), \phi_2(w)) \leq \rho_1 \rho_2 d(z, w)$, for all $z, w \in B_{r_{12}}(x)$.

Thus, $g_2 g_1$ is Ruelle-expanding, which proves our claim.

The previous computations also yield that, if $\rho_i \in (0, 1)$ denotes the backward contraction rate for $g_i \in G_1^*$ and $r_i > 0$ is so that all inverse branches for g_i are defined in balls of radius r_i , then every map $g_{i_2} g_{i_1}$ is Ruelle-expanding and its inverse branches are defined in balls of radius r with backward contraction rates ρ , where

$$r = \min\{r_i : 1 \leq i \leq p\} \quad \text{and} \quad \rho = \min\{\rho_i : 1 \leq i \leq p\}. \quad (12)$$

We now proceed by induction on n . If, for a fixed positive integer n , the concatenation of n Ruelle-expanding maps is Ruelle-expanding, then, considering $n + 1$ such maps, say $g_{i_{n+1}} g_{i_n} \cdots g_{i_1}$, we may split their composition into the concatenation of two Ruelle-expanding maps $g_{i_{n+1}} (g_{i_n} \cdots g_{i_1})$ and apply what we have just proved. This finishes the proof of the first part of the lemma.

We now prove that $S : G \times M \rightarrow M$ is strongly δ -expansive for some $\delta > 0$. Set $\delta = \frac{r}{2}$. Given $\gamma > 0$, let $\kappa_\gamma \geq 1$ be such that $\rho^{\kappa_\gamma} \delta < \gamma$, where ρ is defined by (12). Now, given any

points $x \neq y \in X$ with $d(x, y) \geq \gamma$ and $\underline{g} \in G_\kappa^*$ with $\kappa \geq \kappa_\gamma$, clearly $d_{\underline{g}}(x, y) > \delta$, otherwise,

$$d(x, y) \leq \rho^{\kappa_\gamma} d(\underline{g}(x), \underline{g}(y)) \leq \rho^{\kappa_\gamma} d_{\underline{g}}(x, y) < \gamma$$

which leads to a contradiction. This completes the proof of the lemma. \square

Let us resume the proof of Theorem A. Recall, from Lemma 6.6 and the fact that the semigroup action is strongly expansive, that the computation of the topological entropy of the semigroup action may be done with a well chosen, but fixed, ε . More precisely, if $\delta > 0$ is given by the proof of Lemma 6.9 and $0 < \varepsilon < \delta$ then

$$h_{\text{top}}(S) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right).$$

Fix one such $0 < \varepsilon < \delta$. We claim that for every $\underline{g} \in G$ such that $|\underline{g}| = n$, the set $\text{Fix}(\underline{g})$ is $(\underline{g}, n, \varepsilon)$ -separated. Otherwise, there would exist $P \neq Q \in \text{Fix}(\underline{g})$ which were not $(\underline{g}, n, \varepsilon)$ -separated, that is, such that $d(\underline{g}_m(P), \underline{g}_m(Q)) < \varepsilon$ for every $0 \leq m \leq n$. If $\gamma = d(P, Q)/2$ and $k_\gamma \geq 1$ is given by Lemma 6.9, then $|\underline{g}^{k_\gamma}| = n\kappa_\gamma \geq \kappa_\gamma$ and $d_{\underline{g}^{k_\gamma}}(P, Q) = d_{\underline{g}}(P, Q) < \varepsilon$, which leads to a contradiction. This proves the claim. Therefore, $s(\underline{g}, n, \varepsilon) \geq \#\text{Fix}(\underline{g})$, for every $\underline{g} \in G$ with $|\underline{g}| = n$, and, consequently,

$$\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \geq \frac{1}{p^n} \sum_{|\underline{g}|=n} \#\text{Fix}(\underline{g})$$

and so

$$h_{\text{top}}(S) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{1}{p^n} \sum_{|\underline{g}|=n} \#\text{Fix}(\underline{g}) \right) = \wp(S).$$

To complete the proof we have to show that the limsup in the definition of $\wp(S)$ is indeed a limit. First we notice that, as G is finitely generated by Ruelle-expanding maps, by [24, Theorem 16], it satisfies the periodic orbital specification property. Fix $\varepsilon \in (0, \delta)$, let $T(\varepsilon/2) \in \mathbb{N}$ be given by this property and take $\underline{g} \in G_{m+n+T(\varepsilon/2)}^*$. We observe that there exist $\underline{a} \in G_n$, $\underline{b} \in G_{T(\varepsilon/2)}^*$ and $\underline{c} \in G_m$ such that $\underline{g} = \underline{a} \underline{b} \underline{c}$. Let $\text{Fix}(\underline{c}) = \{P_1, \dots, P_r\}$ and $\text{Fix}(\underline{a}) = \{Q_1, \dots, Q_s\}$ be the sets of fixed points of \underline{c} and \underline{a} , respectively. By the periodic specification property, for the semigroup elements $\underline{c}, \underline{a}$ and the points $P_i \in \text{Fix}(\underline{c})$ and $Q_j \in \text{Fix}(\underline{a})$ there exists $x_{ij} \in \text{Fix}(\underline{a} \underline{b} \underline{c})$ such that

$$d(\underline{c}_\ell(x_{ij}), \underline{c}_\ell(P_i)) < \frac{\varepsilon}{2} \quad \text{and} \quad d(\underline{a}_u \underline{b} \underline{c}(x_{ij}), \underline{a}_u(Q_j)) < \frac{\varepsilon}{2}$$

for every $\ell = 0, \dots, m$ and every $u = 0, \dots, n$. As the set $\text{Fix}(\underline{c})$ is $(\underline{c}, m, \varepsilon)$ -separated, we have $x_{i_1 j_1} \neq x_{i_2 j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$. This implies that $\#\text{Fix}(\underline{g}) \geq \#\text{Fix}(\underline{a}) \#\text{Fix}(\underline{c})$ and so,

$$\sum_{|\underline{g}|=m+n+T(\varepsilon/2)} \#\text{Fix}(\underline{g}) \geq \sum_{|\underline{c}|=m, |\underline{a}|=n} \#\text{Fix}(\underline{c}) \#\text{Fix}(\underline{a}) = \left(\sum_{|\underline{c}|=m} \#\text{Fix}(\underline{c}) \right) \left(\sum_{|\underline{a}|=n} \#\text{Fix}(\underline{a}) \right).$$

This yields

$$\frac{1}{p^{m+n+T(\varepsilon/2)}} \sum_{|\underline{g}|=m+n+T(\varepsilon/2)} \# \text{Fix}(\underline{g}) \geq \frac{1}{p^{T(\varepsilon/2)}} \left(\frac{1}{p^m} \sum_{|\underline{c}|=m} \# \text{Fix}(\underline{c}) \right) \left(\frac{1}{p^n} \sum_{|\underline{a}|=n} \# \text{Fix}(\underline{a}) \right). \quad (13)$$

If we denote by a_n the value $\log \left(\frac{1}{p^n} \sum_{|\underline{a}|=n} \# \text{Fix}(\underline{a}) \right)$, the inequality (13) implies that $a_{m+n+T(\varepsilon/2)} \geq a_n + a_m$ for all $m, n \geq 1$. As $T(\varepsilon/2)$ is a fixed constant, by a simple adaptation of the proof of Fekete's Lemma ([37, Theorem 4.9]), it follows that the sequence $(\frac{a_n}{n})_{n \in \mathbb{N}}$ converges to its supremum. Therefore the lim sup in the definition of $\wp(S)$ may be replaced by a limit. To complete the proof of the Theorem A we are left to show that $\wp(S) = \log sp(\mathbf{L}_{n,0})_n$. By Lemma 6.9, each $\underline{g} \in G$ is a Ruelle-expanding map. Hence, $\# \text{Fix}(\underline{g}) = \deg(\underline{g})$ and then, taking the observable $\varphi \equiv 0$, the equality $\wp(S) = \log sp((\mathbf{L}_{n,0})_n)$ holds trivially by (11).

7. THE ZETA FUNCTION OF THE SEMIGROUP ACTION

Let G be a semigroup generated by a finite set G_1 and $S : G \times M \rightarrow M$ the corresponding continuous semigroup action on a compact connected metric space M .

Definition 7.1. The *zeta function* associated to the continuous semigroup action $S : G \times M \rightarrow M$ is the formal power series

$$z \in \mathbb{C} \mapsto \zeta_S(z) = \exp \left(\sum_{n=1}^{\infty} \frac{N_n(G)}{n} z^n \right), \quad \text{where} \quad N_n(G) = \frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g}). \quad (14)$$

We observe that this notion is a priori different from the one introduced by Artin-Mazur in [2]. In the particular case of G being generated by $G_1^* = \{f\}$, then $G_n^* \subseteq \{f, f^2, \dots, f^n\}$ and we get

$$N_n(G) = \sum_{\{j: f^j \in G_n^*\}} \# \text{Fix}(f^j) \quad \text{and} \quad \zeta_S(z) = \exp \left(\sum_{n=1}^{\infty} \frac{\sum_{\{j: f^j \in G_n^*\}} \# \text{Fix}(f^j)}{n} z^n \right)$$

while the dynamical Artin-Mazur's zeta function computes

$$\zeta_f(z) = \exp \left(\sum_{n=1}^{\infty} \frac{\# \text{Fix}(f^n)}{n} z^n \right).$$

These are different for instance if f is a finite order element in G as a rational rotation on S^1 . However, in the case of a topologically mixing Ruelle-expanding map $f : M \rightarrow M$, we have $G_n^* = \{f^n\}$ and, therefore, the zeta function ζ_S coincides with ζ_f . As mentioned before (cf. Subsection 4), in this context, the periodic points have a definite exponential growth

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Fix}(f^n) = h_{\text{top}}(f) = -\log(\rho_f)$$

where ρ_f is the radius of convergence of the zeta function ζ_f . We refer the reader to [11] for an account on random zeta functions.

7.1. Proof of Theorem B. The function ζ_S we associate to the semigroup G generated by a finite set G_1 , with G_1^* a finite set of Ruelle-expanding maps, is linked to the notion of annealed zeta function introduced in [4] in the context of random families of C^r expanding maps, $r > 1$. The aim of this section is to show that, when we consider Ruelle-expanding maps and the random walk $R_{\underline{p}}$, the annealed zeta function is rational and its radius of convergence is $\exp(-h_{\text{top}}(S))$.

We will start estimating the radius ρ_S of convergence of ζ_S and relating it with $h_{\text{top}}(S)$. We first notice that, as $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$, then

$$\frac{1}{\rho_S} = \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{N_n(G)}{n}} = \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g})}. \quad (15)$$

On the other hand,

$$\limsup_{n \rightarrow \infty} \sqrt[n]{\frac{1}{p^n} \sum_{|\underline{g}|=n} \# \text{Fix}(\underline{g})} = \limsup_{n \rightarrow \infty} \sqrt[n]{\max\{N_n(G), 1\}} = \exp(\varphi(S)).$$

Consequently, $\rho_S = \exp(-\varphi(S))$. Thus, whenever $\varphi(S) > 0$, the zeta function ζ_S has a positive radius of convergence, meaning it is well defined in $\{z \in \mathbb{C} : |z| < \exp(-\varphi(S))\}$. Under the assumptions of Theorem A, one also has $\rho_S = \exp(-h_{\text{top}}(S))$.

We are left to prove that the zeta function of a semigroup of Ruelle-expanding maps is rational. We start showing that, under the assumptions of Theorem B, the skew-product

$$\begin{aligned} \mathcal{F}_G : \Sigma_p^+ \times M &\rightarrow \Sigma_p^+ \times M \\ (\omega, x) &\mapsto (\sigma(\omega), g_{\omega_1}(x)) \end{aligned} \quad (16)$$

where $\omega = (\omega_1, \omega_2, \dots)$, has the following properties.

Lemma 7.2. *The map \mathcal{F}_G is Ruelle-expanding and topologically mixing.*

Proof. Denote by d_M and d_Σ the metrics in M and Σ_p^+ , respectively. We are considering in $\Sigma_p^+ \times M$ the product topology, which is metrizable; its topology is given, for instance, by the metric $D((\omega^0, x_0), (\omega^1, x_1)) = \max\{d_M(x_0, x_1), d_\Sigma(\omega^0, \omega^1)\}$. As σ and each $g_i \in G_1^*$ are Ruelle-expanding, there exist positive constants c_σ and c_i , for $i \in \{1, \dots, p\}$, such that

$$\begin{aligned} x, y \in M, x \neq y, g_i(x) = g_i(y) &\Rightarrow d_M(x, y) > c_i \\ \omega^0, \omega^1 \in \Sigma_p^+, \omega^0 \neq \omega^1, \sigma(\omega^0) = \sigma(\omega^1) &\Rightarrow d_\Sigma(\omega^0, \omega^1) > c_\sigma. \end{aligned}$$

Let $(\omega^0, x_0) \neq (\omega^1, x_1)$ be such that $\mathcal{F}_G((\omega^0, x_0)) = \mathcal{F}_G((\omega^1, x_1))$, that is, $\sigma(\omega^0) = \sigma(\omega^1)$ and $g_{\omega_1^0}(x_0) = g_{\omega_1^1}(x_1)$. Then, either $\omega^0 \neq \omega^1$, in which case we have $D((\omega^0, x_0), (\omega^1, x_1)) > c_\sigma$;

or else $\omega^0 = \omega^1$, and then $D((\omega^0, x_0), (\omega^1, x_1)) > c_{\omega^0}$. Therefore, if $c = \min \{c_\sigma, c_1, c_2, \dots, c_d\}$, then

$$(\omega^0, x_0) \neq (\omega^1, x_1), \mathcal{F}_G((\omega^0, x_0)) = \mathcal{F}_G((\omega^1, x_1)) \Rightarrow D((\omega^0, x_0), (\omega^1, x_1)) > c.$$

Let us now address the second property that characterizes Ruelle-expanding maps. Take r and ρ as in (12), r_σ and ρ_σ the corresponding values for σ due to its expanding nature (see Definition 2.1 and the first example in Subsection 4), and set $r_{\mathcal{F}_G} = \min \{r, r_\sigma\}$ and $\rho_{\mathcal{F}_G} = \min \{\rho, \rho_\sigma\}$. Then, given $(\omega, x) \in \Sigma_p^+ \times M$ and (β, a) such that $\mathcal{F}_G((\beta, a)) = (\omega, x)$, there exist maps $\varphi_{\omega_1} : B_r(x) \rightarrow M$ and $\varphi_\sigma : B_{r_\sigma}(\omega) \rightarrow \Sigma_p^+$ such that $\varphi_{\omega_1}(x) = a$, $\varphi_\sigma(\omega) = \beta$, $g_{\omega_1} \circ \varphi_{\omega_1} = id$, $\sigma \circ \varphi_\sigma = id$ and

$$\begin{aligned} d_M(\varphi_{\omega_1}(z), \varphi_{\omega_1}(w)) &\leq \rho d_M(z, w), \quad \forall z, w \in B_r(x) \\ d_\Sigma(\varphi_\sigma(s), \varphi_\sigma(t)) &\leq \rho_\sigma d_\Sigma(s, t), \quad \forall s, t \in B_{r_\sigma}(\omega). \end{aligned}$$

Therefore, if $R > 0$ is such that, for all $(\gamma, b) \in \Sigma_p^+ \times M$, we have $B_R((\gamma, b)) \subset B_{r_{\mathcal{F}_G}}(\gamma) \times B_{r_{\mathcal{F}_G}}(b)$, then the map $\varphi_\sigma \times \varphi_{\omega_1} : B_R((\omega, x)) \rightarrow \Sigma_p^+ \times M$ satisfies $\varphi_\sigma \times \varphi_{\omega_1}(\omega, x) = (\beta, a)$, $\mathcal{F}_G \circ (\varphi_\sigma \times \varphi_{\omega_1}) = id$ and

$$\begin{aligned} D(\varphi_\sigma \times \varphi_{\omega_1}(s, z), \varphi_\sigma \times \varphi_{\omega_1}(t, w)) &= \max\{d_M(\varphi_{\omega_1}(z), \varphi_{\omega_1}(w)), d_\Sigma(\varphi_\sigma(s), \varphi_\sigma(t))\} \\ &\leq \rho_{\mathcal{F}_G} \max\{d_M(z, w), d_\Sigma(s, t)\} \\ &= \rho_{\mathcal{F}_G} D((s, z), (t, w)) \quad \forall s, t \in B_{r_{\mathcal{F}_G}}(\omega) \times B_{r_{\mathcal{F}_G}}(x). \end{aligned}$$

This ends the proof that \mathcal{F}_G is Ruelle-expanding.

We now proceed showing that \mathcal{F}_G is topologically mixing. Consider a non-empty open subset \mathcal{W} of $\Sigma_p^+ \times M$ and take a cylinder $U = C(1; a_1 a_2 \dots, a_k)$ and an open set V of M such that $U \times V \subset \mathcal{W}$. As the maps σ and $g_{a_k} \dots g_{a_1}$ are topologically mixing and Ruelle-expanding, there exist positive integers m_U and m_V such that $\sigma^\ell(U) = \Sigma_p^+$ and $(g_{a_k} \dots g_{a_1})^\ell = M$, for all $\ell \geq m = \max \{m_U, m_V\}$. Hence, for all $\ell \geq m$ we have

$$\mathcal{F}_G^\ell(U \times V) = \left(\sigma^\ell(U), \bigcup_{\omega \in U} f_\omega^\ell(V) \right) = \Sigma_p^+ \times M$$

since V contains all the sequences of Σ_d^+ whose k first entries are $a_1 a_2 \dots, a_k$, in particular those which start with this block repeated ℓ times for every $\ell \in \mathbb{N}$. \square

Corollary 7.3. *The Artin-Mazur zeta function of \mathcal{F}_G is rational.*

Proof. After Lemma 7.2, we have just to summon Subsection 4. \square

Given $n \in \mathbb{N}$, let $\sharp \text{Per}_n(\mathcal{F}_G)$ be the number of periodic points with period n of \mathcal{F}_G .

Lemma 7.4. $\sharp \text{Per}_n(\mathcal{F}_G) = p^n \times N_n(G)$.

Proof. We observe that $(\omega, x) \in \text{Per}_n(\mathcal{F}_G)$ is and only if $\sigma^n(\omega) = \omega$ and $f_\omega^n(x) = x$. That is, ω is a periodic point with period n of σ , whose set has cardinal p^n , and $x \in \text{Fix}(f_\omega^n)$. Thus

$$\sharp \text{Per}_n(\mathcal{F}_G) = \sum_{\sigma^n(\omega)=\omega} \sharp \text{Fix}(f_\omega^n) = \sum_{|g|=n} \sharp \text{Fix}(g) = p^n \times N_n(G).$$

□

Corollary 7.5. *The zeta function of the semigroup is rational.*

Proof. Let $\zeta_{\mathcal{F}_G}$ be the Artin-Mazur zeta function of the skew-product \mathcal{F}_G . Given $z \in \mathbb{C}$,

$$\begin{aligned}\zeta_S(z) &= \exp \left(\sum_{n=1}^{+\infty} \frac{N_n(G)}{n} z^n \right) = \exp \left(\sum_{n=1}^{+\infty} \frac{\# \text{Per}_n(\mathcal{F}_G)}{n \times p^n} z^n \right) \\ &= \exp \left(\sum_{n=1}^{+\infty} \frac{\# \text{Per}_n(\mathcal{F}_G)}{n} \left(\frac{z}{p} \right)^n \right) = \zeta_{\mathcal{F}_G} \left(\frac{z}{p} \right).\end{aligned}$$

□

8. INTRINSIC OBJECTS *vs* SKEW-PRODUCT DYNAMICS

In this section we will establish a bridge between topological Markov chains and semigroup actions in what concerns the relation between the notions of fibered and relative topological entropies and the concepts of annealed and quenched topological pressures.

8.1. Thermodynamic formalism for the skew-product. There have been several approaches to study the thermodynamic formalism of skew-product dynamics: (i) Ruelle expanding skew-product maps, (ii) fibered entropy for factor maps, (iii) quenched and annealed equilibrium states in random dynamical systems and (iv) relative measures for skew-products. We collect some of them here to be later compared with our results.

Topological entropy of the Ruelle expanding skew-product. Since the skew product \mathcal{F}_G is a Ruelle-expanding map (cf. Lemma 7.2), for any Hölder continuous potential φ on $\Sigma_p^+ \times M$ the map \mathcal{F}_G admits a unique equilibrium state as described in Section 4. In particular, if $\varphi \equiv 0$, there is a unique measure $\mu_{\underline{m}}$ of maximal entropy of \mathcal{F}_G , which is equally distributed along the $\sum_{i=1}^p \deg(g_i)$ elements of the natural Markov partition \mathcal{Q} on $\Sigma_p^+ \times M$ and may be computed by the limit process described in Section 4. From [34], we also know that the projection of $\mu_{\underline{m}}$ in Σ_p^+ is $\eta_{\underline{m}}$, where

$$\underline{m} = \left(\frac{\deg(g_1)}{\sum_{k=1}^p \deg(g_k)}, \frac{\deg(g_2)}{\sum_{k=1}^p \deg(g_k)}, \dots, \frac{\deg(g_p)}{\sum_{k=1}^p \deg(g_k)} \right). \quad (17)$$

Moreover, the topological entropy of the skew-product \mathcal{F}_G is given by

$$h_{\text{top}}(\mathcal{F}_G) = h_{\text{top}}(S) + \log p \quad (18)$$

(cf. [10]), so

$$h_{\text{top}}(\mathcal{F}_G) = \log \left(\sum_{i=1}^p \deg(g_i) \right) \quad \text{and} \quad h_{\text{top}}(S) = \log \left(\frac{\sum_{i=1}^p \deg(g_i)}{p} \right).$$

Notice that the last equality for $h_{\text{top}}(S)$ depends only on the ingredients that set up the semigroup action (in particular, it does not explicitly display the skew-product). More generally (see [28]), if φ is piecewise constant along the $\sum_{i=1}^p \deg(g_i)$ elements of Markov

partition \mathcal{Q} , then there exists a unique equilibrium state μ_φ for \mathcal{F}_G with respect to φ , it is a Bernoulli measure and satisfies

$$P_{\text{top}}(\mathcal{F}_G, \varphi) = \log \left(\sum_{Q \in \mathcal{Q}} e^{\varphi(Q)} \right) \quad (19)$$

and, for every $Q \in \mathcal{Q}$,

$$\mu_\varphi(Q) = \frac{e^{\varphi(Q)}}{\sum_{Q \in \mathcal{Q}} e^{\varphi(Q)}}. \quad (20)$$

Fibered entropy of the skew-product. Following [18], consider the skew-product \mathcal{F}_G and the projection on the first coordinate, say $\pi : \Sigma_p^+ \times M \rightarrow \Sigma_p^+$, $\pi(\omega, x) = \omega$. We say that a subset E of $\pi^{-1}(\omega)$ is (n, ε) -separated if there exists $i \in \{0, \dots, n\}$ such that $d(g_{\omega_i} \dots g_{\omega_1}(x), g_{\omega_i} \dots g_{\omega_1}(y)) \geq \varepsilon$ where $\underline{g} = g_{\omega_n} \dots g_{\omega_1} \in G$ and $|\underline{g}| = n$. Therefore, if $s(n, \varepsilon, \pi^{-1}(\omega))$ is the maximal cardinality of a (n, ε) -separated subset of $\pi^{-1}(\omega)$, then $s(n, \varepsilon, \pi^{-1}(\omega)) = s(\underline{g}, n, \varepsilon)$. By [18], given a σ -invariant probability measure η on Σ_p^+ , then the map

$$\omega \mapsto h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, \varepsilon, \pi^{-1}(\omega))$$

is measurable and

$$\sup_{\{\mu : \mathcal{F}_{G*}(\mu) = \mu, \pi_*(\mu) = \eta\}} h_\mu(\mathcal{F}_G) = h_\eta(\sigma) + \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta(\omega). \quad (21)$$

We will refer to $h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega))$ as the *relative entropy* on the fiber $\pi^{-1}(\omega)$; the *fibered entropy* of the semigroup action S with respect to η is $\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta(\omega)$.

Quenched and annealed equilibrium states for the skew-product. We now follow [4] to define quenched and annealed equilibrium states. Given a continuous potential $\varphi : \Sigma_p^+ \times M \rightarrow \mathbb{R}$ and a probability measure \underline{a} on $\{1, \dots, p\}$, the *annealed topological pressure* of \mathcal{F}_G with respect to φ and \underline{a} is defined as

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu : \mathcal{F}_{G*}\mu = \mu\}} \left\{ h_\mu(\mathcal{F}_G) - h_{\pi_\mu}(\sigma) + h^{\underline{a}}(\pi_\mu) + \int \varphi(\omega, x) d\mu(\omega, x) \right\}$$

where $\omega = (\omega_1, \omega_2, \dots)$, $\pi_\mu = \pi_*(\mu)$ is the marginal of μ in Σ_p^+ and the *entropy per site* $h^{\underline{a}}(\pi_\mu)$ with respect to $\eta_{\underline{a}}$ is given by

$$h^{\eta_{\underline{a}}}(\pi_\mu) = - \int_{\Sigma_p^+} \log \psi_{\pi_\mu}(\omega) d\pi_\mu(\omega) = \int_{\Sigma_p^+} -\psi_{\pi_\mu}(\omega) \log \psi_{\pi_\mu}(\omega) d\underline{a}(\omega_1) d\pi_\mu(\sigma(\omega))$$

if $d\pi_\mu(\omega_1, \omega_2, \dots) \ll d\underline{a}(\omega_1) d\pi_\mu(\omega_2, \omega_3, \dots)$ and $\psi_{\pi_\mu} := \frac{d\pi_\mu}{d\underline{a} d\pi_\mu \circ \sigma}$ denotes the Radon-Nykodin derivative of π_μ with respect to $\underline{a} \times \pi_\mu \circ \sigma$; and $h^{\underline{a}}(\pi_\mu) = -\infty$ otherwise. We recall from [4, page 676] that

$$h^{\underline{a}}(\pi_\mu) = 0 \quad \Leftrightarrow \quad \pi_\mu = \eta_{\underline{a}}.$$

According to [4, Equation (2.28)], the annealed pressure can also be evaluated by

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu: \mathcal{F}_G * \mu = \mu\}} \left\{ h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+ \times M} \log(\underline{a}(\omega_1) e^{\varphi(\omega, x)}) d\mu(\omega, x) \right\}. \quad (22)$$

The *quenched topological pressure* of \mathcal{F}_G with respect to φ and \underline{a} is defined as

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, \varphi, \underline{a}) = \sup_{\{\mu: \mathcal{F}_G * \mu = \mu, \pi_\mu = \eta_{\underline{a}}\}} \left\{ h_\mu(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) + \int \varphi(\omega, x) d\mu(\omega, x) \right\}. \quad (23)$$

It follows from the definition that we always have $P_{\text{top}}^{(a)}(\mathcal{F}_G, \varphi, \underline{a}) \geq P_{\text{top}}^{(q)}(\mathcal{F}_G, \varphi, \underline{a})$.

An \mathcal{F}_G -invariant probability measure is said to be an *annealed (resp. quenched) equilibrium state* for \mathcal{F}_G with respect to φ and \underline{a} if it attains the supremum in equation (22) (resp. equation (23)). In the case of finitely generated semigroups of C^2 expanding maps, there exists a unique quenched and a unique annealed equilibrium state for every Hölder continuous observable φ and every \underline{a} , and they exhibit an exponential decay of correlations (cf. [4]). For instance, for a semigroup G with generators $G_1 = \{id, g_1, \dots, g_p\}$ where each g_i is a C^2 -smooth expanding map, consider the Hölder potential $\varphi(\omega, x) = -\log |\det Dg_{\omega_1}(x)|$. Its annealed equilibrium state $\mu_{\varphi, \underline{a}}^{(a)}$ was described in [4, Proposition 2] and is also the quenched equilibrium state for this potential.

Remark 8.1. Given a continuous potential $\varphi: \Sigma_p^+ \times M \rightarrow \mathbb{R}$ and a σ -invariant probability measure η on Σ_p^+ , the notion of relative pressure of φ on the fiber $\pi^{-1}(\omega)$, denoted by $P_{\text{top}}(\mathcal{F}_G, \varphi, \pi^{-1}(\omega))$, was studied in [18, Section 2]. In this reference it was shown that the relative pressure satisfies the following relative variational principle:

$$\int_{\Sigma_p^+} P_{\text{top}}(\mathcal{F}_G, \varphi, \pi^{-1}(\omega)) d\eta(\omega) = \sup_{\{\mu: \mathcal{F}_G * (\mu) = \mu, \pi_*(\mu) = \eta\}} \left\{ h_\mu(\mathcal{F}_G) - h_\eta(\sigma) + \int \varphi d\mu \right\}. \quad (24)$$

Hence, a quenched equilibrium state for \mathcal{F}_G with respect to φ and \underline{a} is also an \mathcal{F}_G -invariant probability measure $\mu_{\varphi, \underline{a}}^{(q)}$ satisfying

$$\int_{\Sigma_p^+} P_{\text{top}}(\mathcal{F}_G, \varphi, \pi^{-1}(\omega)) d\eta_{\underline{a}}(\omega) = h_{\mu_{\varphi, \underline{a}}^{(q)}}(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) + \int \varphi d\mu_{\varphi, \underline{a}}^{(q)}. \quad (25)$$

Example 8.2. Let G be a semigroup with generators $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 -smooth expanding map. For the potential $\varphi \equiv 0$ we will analyze how the annealed and quenched pressures vary with \underline{a} . When $\underline{a} = \underline{p}$, we obtain

$$\begin{aligned} P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{p}) &= \sup_{\{\mu: \mathcal{F}_G * \mu = \mu\}} \left\{ h_\mu(\mathcal{F}_G) + \int \log(\underline{p}(\omega_1)) d\mu(\omega, x) \right\} \\ &= \sup_{\{\mu: \mathcal{F}_G * \mu = \mu\}} \left\{ h_\mu(\mathcal{F}_G) - \log p \right\} = h_{\text{top}}(\mathcal{F}_G) - \log p = h_{\text{top}}(S). \end{aligned} \quad (26)$$

The corresponding annealed equilibrium state $\mu_{\underline{p}}^{(a)}$ is the measure of maximal entropy μ_m of \mathcal{F}_G . Moreover, by [4, Equation (2.28)], for any non-trivial probability vector \underline{a} and the

corresponding annealed equilibrium state $\mu_{\underline{a}}^{(a)}$ we have

$$h_{\pi_{\mu_{\underline{a}}^{(a)}}}(\sigma) = h_{\pi_{\mu_{\underline{a}}^{(a)}}}(\sigma) + \int_{\Sigma_p^+ \times M} \log(\underline{a}(\omega_1)) d\mu_{\underline{a}}^{(a)}(\omega, x).$$

So, when $\underline{a} = \underline{p} = (\frac{1}{p}, \dots, \frac{1}{p})$,

$$h_{\pi_{\mu_{\underline{p}}^{(a)}}}(\sigma) = h_{\pi_{\mu_{\underline{p}}^{(a)}}}(\sigma) + \int_{\Sigma_p^+ \times M} \log(\underline{p}(\omega_1)) d\mu_{\underline{p}}^{(a)}(\omega, x) = h_{\eta_{\underline{m}}}(\sigma) - \log p = h_{\eta_{\underline{m}}}(\sigma) - h_{\eta_{\underline{p}}}(\sigma). \quad (27)$$

Concerning the quenched operator, we get

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{p}) = \sup_{\{\mu: \mathcal{F}_G * \mu = \mu, \pi_\mu = \eta_{\underline{p}}\}} \left\{ h_\mu(\mathcal{F}_G) \right\} - \log p.$$

Yet, as $\pi(\mu_{\underline{m}}) = \eta_{\underline{m}}$ and $\underline{m} \neq \underline{p}$ (cf. [34]), the quenched equilibrium state differs from $\mu_{\underline{p}}^{(a)}$. In general, for a non-trivial probability vector \underline{a} , we have

$$\begin{aligned} P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}) &= \sup_{\{\mu: \mathcal{F}_G * \mu = \mu\}} \left\{ h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+ \times M} \log(\underline{a}(\omega_1)) d\mu(\omega, x) \right\} \\ &= P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{a}}) \end{aligned} \quad (28)$$

where $\varphi_{\underline{a}}: \Sigma_p^+ \times M \rightarrow \mathbb{R}$ is the locally constant potential given by

$$\varphi_{\underline{a}}(\omega, x) = \log \underline{a}(\omega_0)$$

and the quenched pressure is given by

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{a}) = \sup_{\{\mu: \mathcal{F}_G * \mu = \mu, \pi_\mu = \eta_{\underline{a}}\}} \left\{ h_\mu(\mathcal{F}_G) \right\} - h_{\eta_{\underline{a}}}(\sigma).$$

Therefore, a quenched equilibrium state $\mu_{\underline{a}}^{(q)}$ for $\varphi \equiv 0$ and \underline{a} satisfies $\pi_*(\mu_{\underline{a}}^{(q)}) = \eta_{\underline{a}}$ and

$$h_{\mu_{\underline{a}}^{(q)}}(\mathcal{F}_G) = \sup_{\{\mu: \mathcal{F}_G * \mu = \mu, \pi_\mu = \eta_{\underline{a}}\}} h_\mu(\mathcal{F}_G). \quad (29)$$

In particular, when $\underline{a} = \underline{m}$, we conclude that

$$\mu_{\underline{m}}^{(q)} = \mu_{\underline{m}} = \mu_{\underline{p}}^{(a)} \quad \text{and} \quad P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{m}) = h_{\text{top}}(\mathcal{F}_G) - h_{\eta_{\underline{m}}}(\sigma). \quad (30)$$

Relative measures for the skew-product. Another proposal to study skew-product dynamics and semigroup actions was explored in [34, 35]. In [34, Theorem 1.3] it is shown that, if $\eta_{\underline{a}}$ is the Bernoulli probability measure on Σ_p^+ determined by the vector $\underline{a} = (a_1, a_2, \dots, a_p)$, such that $a_k > 0$ for all $k = 1, \dots, p$ and $\sum_{k=1}^p a_k = 1$, there exists a self-similar probability measure $\mu_{\underline{a}}$ which is invariant under the skew-product \mathcal{F}_G , whose projection to the base space is $\pi_*(\mu_{\underline{a}}) = \eta_{\underline{a}}$ and which satisfies

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = \sup_{\{\mu: \mathcal{F}_G * (\mu) = \mu, \pi_*(\mu) = \eta_{\underline{a}}\}} h_\mu(\mathcal{F}_G) \quad (31)$$

and

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = h_{\eta_{\underline{a}}}(\sigma) + \sum_{k=1}^p a_k \log \deg(g_k). \quad (32)$$

where $\deg(g_k)$ stands for the degree of the map g_k . In the particular case of $\underline{a} = \underline{m}$, the measure $\mu_{\underline{m}}$ is the unique probability of maximal entropy of the skew-product \mathcal{F}_G (cf. Subsection 8.1), and a simple computation yields that

$$h_{\text{top}}(\mathcal{F}_G) = \log \left(\sum_{i=1}^p \deg(g_i) \right). \quad (33)$$

8.2. Topological entropy of the semigroup action with respect to a random walk. In what follows we compare the notions of entropy for the semigroup action with the previous notions of fibered, quenched, annealed and relative pressures.

Fibered entropy for the symmetric random walk. We first study the fibered entropy in the case of a symmetric random walk, that is, when each semigroup generator receives the same weight $\frac{1}{p}$. Notice that, in the case of a Bernoulli measure $\eta = \eta_{\underline{a}}$, the relations (23) and (21) imply that the fibered entropy can be computed as a quenched pressure by

$$\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{a}}(\omega) = P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{a}).$$

If $\eta = \eta_{\underline{p}}$, the formula (21) becomes

$$\sup_{\{\mu: \mathcal{F}_{G*}(\mu) = \mu, \pi_*(\mu) = \eta_{\underline{p}}\}} h_{\mu}(\mathcal{F}_G) = \log p + \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}}(\omega). \quad (34)$$

Thus, taking into account that $h_{\text{top}}(\mathcal{F}_G) = \sup_{\mu} h_{\mu}(\mathcal{F}_G)$ (cf. [16]), we conclude that

$$\log p + \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}} \leq h_{\text{top}}(\mathcal{F}_G) \leq \log p + \sup_{\omega \in \Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)). \quad (35)$$

Besides, (18) indicates that $h_{\text{top}}(\mathcal{F}_G) = h_{\text{top}}(S) + \log p$. Together with (35), this implies

$$\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}} \leq h_{\text{top}}(S) \leq \sup_{\omega \in \Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)).$$

So, it makes sense to ask under what conditions we have $h_{\text{top}}(S) = \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}}$.

Proposition 8.3. *Let $G_1 = \{id, g_1, \dots, g_p\}$, $p \geq 2$, be a finite set of expanding maps in $\text{End}^2(M)$, G be the semigroup generated by G_1 and $\mathcal{F}_G : \Sigma_p^+ \times M \rightarrow \Sigma_p^+ \times M$ be the corresponding skew-product. Then the following conditions are equivalent:*

- (1) $h_{\text{top}}(S) = \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}}$.
- (2) $\eta_{\underline{p}} = \pi_*(\mu_{\underline{m}})$, where $\mu_{\underline{m}}$ is the unique maximal entropy measure for \mathcal{F}_G .
- (3) The degrees of the maps g_i are the same for all $1 \leq i \leq p$.

Moreover, if any of these conditions holds, then

$$h_{\text{top}}(S) = \int_{\Sigma_p^+} \log \deg(g_\omega) d\eta_{\underline{p}}(\omega).$$

Proof. Recall that \mathcal{F}_G is a topologically mixing Ruelle-expanding map (cf. Lemma 7.2). Hence, taking $\varphi \equiv 0$ in Theorem 4.1, we deduce that \mathcal{F}_G has a unique maximal entropy measure $\mu_{\underline{m}}$. Furthermore, it follows from the variational principle for \mathcal{F}_G and the relations (34) and (18) that

$$\begin{aligned} h_{\text{top}}(S) = h_{\text{top}}(\mathcal{F}_G) - \log p &\geq \sup_{\{\mu: \mathcal{F}_{G*}(\mu) = \mu, \pi_*(\mu) = \eta_{\underline{p}}\}} [h_\mu(\mathcal{F}_G) - \log p] \\ &= \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}}(\omega). \end{aligned}$$

Clearly the previous inequality becomes an equality if and only if $\pi_*(\mu_{\underline{m}}) = \eta_{\underline{p}}$, which proves that (1) is equivalent to (2). The remaining of the proof relies on the property of $\mu_{\underline{m}}$ stated in (17). It implies that $h_{\text{top}}(S) = \int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{p}}$ if and only if the degrees of the maps g_i are the same for all $1 \leq i \leq p$. This proves that (2) is equivalent to (3). Finally, as $\eta_{\underline{p}}$ is a Bernoulli measure, $h_{\eta_{\underline{p}}}(\sigma) = \log p$ and $h_{\mu_{\underline{m}}}(\mathcal{F}_G) = h_{\text{top}}(\mathcal{F}_G)$, then

$$h_{\text{top}}(S) = \frac{1}{p} \sum_{k=1}^p \log \deg(g_k) = \int_{\Sigma_p^+} \log \deg(g_{\omega_1}) d\eta_{\underline{p}}(\omega).$$

This completes the proof of the proposition. \square

Relative entropy for non-symmetric random walks. The notion $h_{\text{top}}(S)$ of topological entropy of a semigroup action S depends on our choice of the Bernoulli equally distributed probability measure $\eta_{\underline{p}}$ on Σ_p^+ . If, instead of $\eta_{\underline{p}}$, we take another σ -invariant Bernoulli probability measure η on the Borel sets of Σ_p^+ , then the measure η portrays an asymmetric random walk on G and it suggests a generalization of the concept of topological entropy of S .

Definition 8.4. Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 expanding map on a compact connected Riemannian manifold M , and let $S : G \times M \rightarrow M$ be the corresponding continuous semigroup action. Consider a probability measure η on Σ_p^+ . The *relative topological entropy of the semigroup action S with respect to η* is given by

$$h_{\text{top}}(S, \eta) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_{\Sigma_p^+} s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) d\eta(\omega)$$

where $s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ denotes the maximum cardinality of a $(\underline{g}, n, \epsilon)$ -separated set (see Section 6) and $\omega = \omega_1 \omega_2 \dots \omega_n \dots$.

The previous notion is well defined since the map $\omega \rightarrow s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is constant on n -cylinders (hence measurable), bounded by $e^{n \max_{i \in \{1, \dots, p\}} \{h_{\text{top}}(g_i)\}}$ and $s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is

monotonic in the variable ϵ . For instance, $h_{\text{top}}(S, \eta_p) = h_{\text{top}}(S)$. In view of Definition 8.4, $h_{\text{top}}(S, \eta)$ is also given by

$$h_{\text{top}}(S, \eta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{|\underline{g}|=n} \iota_*(\eta)(\underline{g}) \mathcal{L}_{\underline{g},0}(\mathbf{1}) \right).$$

Notice that this formula makes sense since $\underline{g} \mapsto \mathcal{L}_{\underline{g},0}(\mathbf{1})$ is bounded, its values are away from 0 and ∞ and $\iota_*(\eta)$ is a probability measure.

Following an argument analogous to the one used to prove [24, Theorem 25], we obtain:

Corollary 8.5. *Assume that the continuous action of G on the compact metric space M is strongly δ^* -expansive and let $\eta \in \mathcal{M}(\Sigma_p^+)$ be a σ -invariant probability measure. Then, for every $0 < \epsilon < \delta^*$, we have*

$$h_{\text{top}}(\eta, S) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int_{\Sigma_p^+} s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) d\eta(\omega).$$

A variational principle for the relative entropy. Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 expanding map on a compact connected Riemannian manifold M , and let $S : G \times M \rightarrow M$ be the corresponding continuous semigroup action. Note that the action of G on the compact metric space M is strongly δ^* -expansive. Denote by $\mathcal{M}_B(\Sigma_p^+)$ the space of Bernoulli measures on Σ_p^+ , that is, the probability measures $\eta = \eta_{\underline{a}}$ for some probability vector $\underline{a} = (a_1, \dots, a_p)$, where some of the entries may be zero. This space of σ -invariant measures, which encodes all random walks on the semigroup G we have considered so far, is homeomorphic to the finite dimensional simplex $\{\underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p : a_i \geq 0 \text{ and } \sum_{i=1}^p a_i = 1\}$, and therefore it is a closed subset of $\mathcal{M}_\sigma(\Sigma_p^+)$. Let \mathcal{H} be the entropy map with respect to the random walks in $\mathcal{M}_B(\Sigma_p^+)$, given by

$$\begin{aligned} \mathcal{H} : \mathcal{M}_B(\Sigma_p^+) &\rightarrow [0, +\infty] \\ \eta &\mapsto h_{\text{top}}(S, \eta). \end{aligned}$$

Lemma 8.6. *The map \mathcal{H} is continuous.*

Proof. To prove that \mathcal{H} is lower semicontinuous, we go back to the proof of Theorem A where it was established that, if $\epsilon \in (0, \delta^*)$, $T(\epsilon/2) \in \mathbb{N}$ is given by the periodic specification property and $\underline{g} = g_{\omega_{m+n+T(\epsilon/2)}} \dots g_{\omega_1} \in G_{m+n+T(\epsilon/2)}^*$, then there exist

$$\begin{aligned} \underline{a} &= g_{\omega_{m+n+T(\epsilon/2)}} \dots g_{\omega_{m+T(\epsilon/2)+1}} \in G_n^* \\ \underline{b} &= g_{\omega_{m+T(\epsilon/2)}} \dots g_{\omega_{m+1}} \in G_{T(\epsilon/2)}^* \\ \underline{c} &= g_{\omega_m} \dots g_{\omega_1} \in G_m^* \end{aligned}$$

such that $\underline{g} = \underline{a} \underline{b} \underline{c}$ and $\# \text{Fix}(\underline{g}) \geq \# \text{Fix}(\underline{a}) \# \text{Fix}(\underline{c})$, and so

$$\int \# \text{Fix}(\underline{g}) d\eta \geq \int \# \text{Fix}(\underline{c}) d\eta \int \# \text{Fix}(\underline{a}) d\eta.$$

In particular, as we are restricting to $\mathcal{M}_B(\Sigma_p^+)$, we get

$$\mathcal{H}(\eta) = \sup_{n \geq 1} \frac{1}{n} \log \int \# \text{Fix}(g_{\omega_n} \dots g_{\omega_1}) d\eta(\omega) \quad (36)$$

is the supremum of the continuous functions $\eta \mapsto \frac{1}{n} \log \int \# \text{Fix}(g_{\omega_n} \dots g_{\omega_1}) d\eta$, hence lower semicontinuous.

We proceed by showing that \mathcal{H} is also upper semicontinuous. This is due to the characterization of the topological entropy via generating sets. Indeed, the same steps of the proof of Theorem 25 of [24] (where integration is considered there with respect to the equidistributed Bernoulli measure η_p) imply that, as S is a strongly δ^* -expansive continuous semigroup action, then the topological entropy $h_{\text{top}}(S, \eta)$ satisfies

$$h_{\text{top}}(S, \eta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_{\Sigma_p^+} c(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) d\eta(\omega)$$

for every $0 < \epsilon < \delta^*$, where

$$c(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) = \inf \{ \# \mathcal{U} : \mathcal{U} \text{ is a } (g_{\omega_n} \dots g_{\omega_1}, \epsilon) - \text{cover} \}.$$

Take $\epsilon > 0$, $n \in \mathbb{N}$ and $\underline{g} = g_{\omega_{n+m}} \dots g_{\omega_1} \in G$, and consider $\underline{\ell} = g_{\omega_{n+m}} \dots g_{\omega_{n+1}}$ and $\underline{k} = g_{\omega_n} \dots g_{\omega_1}$. Given a $(\underline{\ell}, \epsilon)$ -cover \mathcal{U} and a $(\underline{k}, \epsilon)$ -cover \mathcal{V} , we have that $\mathcal{W} = \underline{k}^{-1}(\mathcal{U}) \vee \mathcal{V}$ is a $(\underline{g}, \epsilon)$ -cover with $\# \mathcal{W} \leq \# \mathcal{U} \# \mathcal{V}$. This implies that

$$c(g_{\omega_{n+m}} \dots g_{\omega_1}, n+m, \epsilon) \leq c(g_{\omega_{n+m}} \dots g_{\omega_{n+1}}, m, \epsilon) c(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon).$$

Since all measures $\eta \in \mathcal{M}_B(\Sigma_p^+)$ are Bernoulli, the previous inequality yields

$$\int_{\Sigma_p^+} c(g_{\omega_{n+m}} \dots g_{\omega_n}, n, \epsilon) d\eta(\omega) \leq \int_{\Sigma_p^+} c(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) d\eta(\omega) \int_{\Sigma_p^+} c(g_{\omega_m} \dots g_{\omega_1}, m, \epsilon) d\eta(\omega).$$

Hence, the sequence $(a_n)_{n \in \mathbb{N}} = \left(\log \int_{\Sigma_p^+} c(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon) d\eta(\omega) \right)_{n \in \mathbb{N}}$ is subadditive and so $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \rightarrow \infty} \frac{a_n}{n}$. In other words,

$$\mathcal{H}(\eta) = \inf_{n \geq 1} \frac{1}{n} \log \int c(g_{\omega_n} \dots g_{\omega_1}) d\eta(\omega)$$

is the infimum of the continuous functions $\eta \mapsto \frac{1}{n} \log \int c(g_{\omega_n} \dots g_{\omega_1}) d\eta(\omega)$. Therefore, \mathcal{H} is upper semicontinuous. \square

Proposition 8.7. *There exists $\eta_0 \in \mathcal{M}_B(\Sigma_p^+)$ such that*

$$\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) = h_{\text{top}}(S, \eta_0).$$

Moreover,

$$\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) = \log \left(\max_{1 \leq i \leq p} \deg(g_i) \right).$$

And $\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) = h_{\text{top}}(S, \eta_p)$ if and only if $\deg(g_i) = d$ for every $1 \leq i \leq p$.

Proof. The first assertion is a direct consequence of the compactness of $\mathcal{M}_B(\Sigma_p^+)$ together with the continuity of the function \mathcal{H} . We are left to prove that

$$\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) = \log \left(\max_{1 \leq i \leq p} \deg(g_i) \right). \quad (37)$$

Let $1 \leq j \leq p$ be such that $\deg(g_j) = \max_{1 \leq i \leq p} \deg(g_i)$. Take $\underline{a} = (a_i)_{1 \leq i \leq p}$, where $a_i = \delta_{ij}$ is the Kronecker delta function, and $\eta_{\underline{a}} = \delta_{jjj\dots}$. Then

$$\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) \geq h_{\text{top}}(S, \eta_{\underline{a}}) = h_{\text{top}}(g_j) = \log \deg(g_j) = \log \left(\max_{1 \leq i \leq p} \deg(g_i) \right).$$

Conversely, assume that $\sup_{\eta \in \mathcal{M}_B(\Sigma_p^+)} h_{\text{top}}(S, \eta) > \log \left(\max_{1 \leq i \leq p} \deg(g_i) \right)$. Using (36), we may find $\delta > 0$ and $\eta \in \mathcal{M}_B(\Sigma_p^+)$ satisfying

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \int \# \text{Fix}(g_{\omega_n} \dots g_{\omega_1}) d\eta(\omega) > \log \left(\max_{1 \leq i \leq p} \deg(g_i) \right) + 2\delta.$$

Then, for every sufficiently large n , there exists $\omega \in \Sigma_p^+$, depending on n , such that

$$\# \text{Fix}(g_{\omega_n} \dots g_{\omega_1}) > e^{\delta n} \left(\max_{1 \leq i \leq p} \deg(g_i) \right)^n$$

which contradicts the growth rate of the fixed point sets of the expanding maps $g_{\omega_n} \dots g_{\omega_1}$. Finally, recall that

$$h_{\text{top}}(S, \eta_p) = \log \left(\frac{\sum_{i=1}^p \deg(g_i)}{p} \right)$$

so η_p is a maximizing measure for \mathcal{H} if and only if the maps g_i , for $1 \leq i \leq p$, have equal degrees. \square

Remark 8.8. It is clear from the equality in (37) that any probability measure in $\mathcal{M}_B(\Sigma_p^+)$ that attains the maximum of \mathcal{H} is of the form $\eta_{\underline{a}} \in \mathcal{M}_B(\Sigma_p^+)$ for some \underline{a} in the simplex

$$S = \left\{ \underline{a} = (a_1, \dots, a_p) \in \mathbb{R}_0^+ : \sum_{i=1}^p a_i = 1 \text{ and } a_j = 0 \text{ whenever } \deg(g_j) \neq \max_{1 \leq i \leq p} \deg(g_i) \right\}.$$

In particular, uniqueness of such maximizing measures holds if and only if there exists a unique expanding map with larger degree.

Relative entropy vs. annealed and quenched pressure. In order to compare the relative entropies of the semigroup action with the several notions of pressure for skew-product dynamics we shall use the transfer operators defined in Section 5. As previously remarked, when $\eta = \eta_{\underline{a}}$, we have

$$h_{\text{top}}(S, \eta_{\underline{a}}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\tilde{\mathbf{L}}_{\underline{a}, \varphi}^n(\mathbf{1})\|_0 = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\hat{\mathbf{L}}_{\underline{a}, \varphi}^n(\mathbf{1})\|_0.$$

It follows from [4, Proposition 3.2] that the spectral radius of $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ coincides with the term $\exp(-P_{\text{top}}^{(a)}(\mathcal{F}_G, \underline{\varphi}, \underline{a}))$ and, for that reason,

$$h_{\text{top}}(S, \eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}). \quad (38)$$

This means that the relative entropy for asymmetric random walks coincides with the annealed topological pressure and also implies the following generalization of formula (18).

Corollary 8.9. *Let $G_1 = \{id, g_1, \dots, g_p\}$, $p \geq 2$, be a finite set of expanding maps in $\text{End}^2(M)$. Given a non-trivial probability vector $\underline{a} = (a_1, a_2, \dots, a_p)$, consider the Bernoulli probability measure $\eta_{\underline{a}}$ on Σ_p^+ and the annealed equilibrium state $\mu_{\underline{a}}^{(a)}$ for \mathcal{F}_G with respect to $\varphi \equiv 0$ and \underline{a} . Then $h_{\text{top}}(S, \eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a})$. If $h^{\underline{a}}(\pi_{\mu_{\underline{a}}^{(a)}}) = h_{\pi_{\mu_{\underline{a}}^{(a)}}}(\sigma) - h_{\eta_{\underline{a}}}(\sigma)$, then*

$$h_{\mu_{\underline{a}}^{(a)}}(\mathcal{F}_G) = h_{\text{top}}(S, \eta_{\underline{a}}) + h_{\eta_{\underline{a}}}(\sigma).$$

Proof. This is a direct consequence of the equality (38) since, under the assumption on $h^{\underline{a}}(\pi_{\mu_{\underline{a}}^{(a)}})$, we have

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}) = \sup_{\{\mu: \mathcal{F}_{G*}\mu = \mu\}} \left\{ h_{\mu}(\mathcal{F}_G) - h_{\pi_{\mu}}(\sigma) + h^{\underline{a}}(\pi_{\mu}) \right\} = h_{\mu_{\underline{a}}^{(a)}}(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma).$$

Notice that the condition on $h^{\underline{a}}(\pi_{\mu_{\underline{a}}^{(a)}})$ is fulfilled when $\underline{a} = \underline{p}$ (cf. (27)). \square

Given a non-trivial probability vector \underline{a} , for some potentials φ the transfer operator $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ coincides with the averaged normalized transfer operator used in [34]. Therefore, we may match the values of the corresponding pressures and their equilibrium states, and deduce the following thermodynamic criterium for the self-similar probability measures constructed in [34].

Proposition 8.10. *Let $G_1 = \{id, g_1, \dots, g_p\}$, $p \geq 2$, be a finite set of expanding maps in $\text{End}^2(M)$. Consider the semigroup G generated by G_1 and the corresponding skew-product $\mathcal{F}_G : \Sigma_p^+ \times M \rightarrow \Sigma_p^+ \times M$. If $\underline{a} = (a_1, a_2, \dots, a_p)$ is a non-trivial probability vector, $\eta_{\underline{a}}$ the Bernoulli probability measure on Σ_p^+ determined by \underline{a} and $\mu_{\underline{a}}$ the self-similar probability measure constructed in [34], then the following assertions are equivalent:*

- (1) $h_{\mu_{\underline{a}}}(\mathcal{F}_G) = \sup_{\{\mu: \mathcal{F}_{G*}(\mu) = \mu, \pi_*(\mu) = \eta_{\underline{a}}\}} h_{\mu}(\mathcal{F}_G)$.
- (2) $P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{a}) = \int \log \deg(g_i) d\underline{a}(i)$.

We observe that, by (29), the condition (1) is equivalent to say that $\mu_{\underline{a}} = \mu_{\underline{a}}^{(q)}$, where $\mu_{\underline{a}}^{(q)}$ is the unique quenched equilibrium state of \mathcal{F}_G with respect to $\varphi \equiv 0$ and \underline{a} .

Proof. Fix $\underline{a} = (a_1, a_2, \dots, a_p)$ and the potential $\underline{\varphi} = (-\log \deg(g_1), \dots, -\log \deg(g_p))$. The transfer operator $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ is precisely the averaged normalized transfer operator introduced in [34]. Therefore, $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}} 1 = 1$, and consequently $P_{\text{top}}^{(a)}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) = 0$. Moreover, $\mu_{\underline{a}} = \mu_{\varphi, \underline{a}}^{(a)}$,

which is the unique annealed equilibrium state for \mathcal{F}_G with respect to $\underline{\varphi}$ and \underline{a} . So, equation (22) yields

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = - \int \log(\underline{a}(\omega_1) e^{\varphi(\omega, x)}) d\mu_{\underline{a}}.$$

On the other hand, the quenched variational principle indicates that

$$\sup_{\{\mu: \mathcal{F}_G * \mu = \mu, \pi_\mu = \eta_{\underline{a}}\}} \left\{ h_\mu(\mathcal{F}_G) \right\} - h_{\eta_{\underline{a}}}(\sigma) = P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{a}).$$

Thus the condition (1) in the statement of the proposition is equivalent to

$$\begin{aligned} P_{\text{top}}^{(q)}(\mathcal{F}_G, 0, \underline{a}) &= - \int \log(\underline{a}(\omega_1) e^{\varphi(\omega, x)}) d\mu_{\underline{a}} - h_{\eta_{\underline{a}}}(\sigma) \\ &= \sum_{i=1}^p -a_i \log a_i + \sum_{i=1}^p a_i \log \deg(g_i) + \sum_{i=1}^p a_i \log a_i = \int \log \deg(g_i) d\underline{a}(i). \end{aligned}$$

□

Corollary 8.11. *Let $G_1^* = \{g_1, \dots, g_p\}$, $p \geq 2$, be a finite set of C^2 expanding maps, G the semigroup generated by G_1 and $\mathcal{F}_G : \Sigma_p^+ \times M \rightarrow \Sigma_p^+ \times M$ the corresponding skew-product. Given a non-trivial probability vector \underline{a} and the Bernoulli probability measure $\eta_{\underline{a}}$, we have*

$$\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_a(\omega) = \sum_{k=1}^p a_k \log \deg(g_k) = \int_{\Sigma_p^+} \log \deg(g_{\omega_1}) d\eta_a(\omega).$$

Proof. We start noticing that, for any non-trivial probability vector \underline{a} and its self-similar probability measure $\mu_{\underline{a}}$, the condition (1) in the statement of Proposition 8.10 is valid (cf. (31)). Therefore, using Proposition 8.10, the equations (32) and (21) yield

$$\int_{\Sigma_p^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_a = \sum_{k=1}^p a_k \log \deg(g_k).$$

The second equality in the statement is an immediate consequence from two facts: in each cylinder $C(1; k) := \{\omega \in \Sigma_p^+ : \omega_1 = k\}$, the map $\log \deg(g_{\omega_1})$ is constant; and $\eta_a(C(1; k)) = a_k$ for any $k \in \{1, 2, \dots, p\}$. □

8.3. Examples. Let us analyze two (non-abelian) examples that illustrate the range of applications of our results on semigroup actions, transfer operators and the dynamical zeta function.

Example 8.12. Let $g_1, g_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the circle expanding maps given by $g_1(z) = z^2$ and $g_2(z) = z^3$ and consider the semigroup G generated by $G_1 = \{id, g_1, g_2\}$. A simple computation shows that, for every $n \in \mathbb{N}$

$$N_n(G) = \frac{1}{2^n} \sum_{k=0}^n (2^k 3^{n-k} - 1) = \mathcal{O} \left(\frac{5}{2} \right)^n$$

and, consequently, $\wp(S) = \log(\frac{5}{2})$ and $\rho_S = \frac{2}{5}$. Moreover, it follows from (18) that

$$h_{\text{top}}(S) = \log 5 - \log 2.$$

If $\eta_{\underline{m}}$ is the Bernoulli probability measure on Σ_2^+ determined by the weights $\underline{m} = (\frac{2}{5}, \frac{3}{5}) \equiv (0.4, 0.6)$, equation (32) becomes

$$h_{\mu_{\underline{m}}}(\mathcal{F}_G) = h_{\text{top}}(\mathcal{F}_G) = \log 5$$

and equation (21) informs that

$$\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{m}}(\omega) = \frac{2}{5} \log 2 + \frac{3}{5} \log 3.$$

So,

$$h_{\text{top}}(S) < \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{m}}(\omega).$$

Concerning $\underline{a} = (\frac{1}{2}, \frac{1}{2}) \equiv (0.5, 0.5)$, which corresponds to the probability measure $\eta_{\underline{a}} = \eta_{\underline{2}}$ on Σ_2^+ , we deduce from Proposition 8.3 that

$$h_{\text{top}}(S) > \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{2}}(\omega).$$

We may add that, if $\mu_{\underline{a}}$ is the self-similar measure previously mentioned that is assigned to \underline{a} in [34], then, again by (32) and (21), we have

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = \log 2 + \frac{\log 2 + \log 3}{2}$$

and

$$\sup_{\mu: \mathcal{F}_{G*}(\mu) = \mu, \pi_*(\mu) = \eta_{\underline{2}}} h_{\mu}(\mathcal{F}_G) = \log 2 + \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{2}}(\omega).$$

Consequently,

$$\frac{\log 2 + \log 3}{2} \leq \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{2}}(\omega) < h_{\text{top}}(S) = \log 5 - \log 2.$$

Moreover, from Corollary 8.11, we conclude that

$$\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{2}}(\omega) = \frac{\log 2 + \log 3}{2}$$

and, more generally, that, for any choice of $\underline{a} = (a_1, a_2)$ with $a_i > 0$ and $a_1 + a_2 = 1$, we have

$$\int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{a}}(\omega) = a_1 \log 2 + a_2 \log 3.$$

Therefore, there is a (unique) vector \underline{a} whose corresponding probability measure $\eta_{\underline{a}}$ on Σ_2^+ satisfies

$$h_{\text{top}}(S) = \int_{\Sigma_2^+} h_{\text{top}}(\mathcal{F}_G, \pi^{-1}(\omega)) d\eta_{\underline{a}}(\omega),$$

namely

$$\underline{a} = \left(\frac{\log \frac{6}{5}}{\log \frac{3}{2}}, \frac{\log \frac{5}{4}}{\log \frac{3}{2}} \right) \approx (0.45, 0.55).$$

Example 8.13. Given $p \in \mathbb{N}$, let $A_i \in GL(p, \mathbb{Z})$ induce linear expanding endomorphisms $g_i := g_{A_i}$ on \mathbb{T}^p , for $i = 1, \dots, p$. Consider the continuous potential $\varphi : \Sigma_p^+ \times M \rightarrow \mathbb{R}$ given by $\varphi(\omega, x) = -\log |\det Dg_{\omega_1}(x)| = -\log \deg(g_{\omega_1})$. Then, by [4, Proposition 2], the quenched and the annealed equilibrium states of φ coincide and are SRB measures. In this setting, for any non-trivial probability vector \underline{a} , the self-similar \mathcal{F}_G -invariant probability measure $\mu_{\underline{a}}$ constructed in [34] coincides with the annealed (hence quenched) SRB measure for \mathcal{F}_G with respect to \mathcal{F} , $\underline{\varphi}$ and \underline{a} . In particular, we have

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) = h_{\mu_{\underline{a}}}(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) + \int \varphi d\mu_{\underline{a}}$$

that is,

$$h_{\mu_{\underline{a}}}(\mathcal{F}_G) = P_{\text{top}}^{(q)}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) + h_{\eta_{\underline{a}}}(\sigma) - \int \varphi d\mu_{\underline{a}}$$

so, comparing this equality with (32), we obtain

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, \underline{\varphi}, \underline{a}) = \sum_{i=1}^p a_i \log \deg(g_i) - \int \varphi d\mu_{\underline{a}} = \int \log \deg(g_i) d\underline{a}(i) - \int \varphi d\mu_{\underline{a}}.$$

9. STATIONARY MEASURES

Since probability measures invariant under all elements of the semigroup are unlikely to exist, the concept of stationary measure is the most natural to be addressed while studying ergodic properties of semigroup actions. Consider a finite set $G_1 = \{id, g_1, \dots, g_p\}$, $p \geq 2$, of C^2 expanding maps on a compact connected Riemannian manifold M and let G be the semigroup generated by G_1 . Denote by (Σ_p^+, σ) the full shift in p symbols from the alphabet $\{1, \dots, p\}$, by \underline{p} the vector $(\frac{1}{p}, \dots, \frac{1}{p})$ and by $\eta_{\underline{p}}$ the equally distributed Bernoulli σ -invariant probability measure on the Borel sets of Σ_p^+ given by the product measure of $\theta(\{i\}) = \frac{1}{p}$ for any $i \in \{1, \dots, p\}$.

Given a point $x \in M$ and $\omega = (\omega_1, \omega_2, \dots) \in \Sigma_p^+$, we define the random orbit of x as

$$x, \quad g_{\omega_1}(x), \quad g_{\omega_2} g_{\omega_1}(x), \dots, g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1}(x), \dots$$

This means that, for each $x \in M$ and $n \in \mathbb{N}$, the evolution of x up to time n is described by its images under the maps obtained by the concatenation of the g_{ω_i} 's up to ω_n . In what follows we will refer to this map as $f_{\omega}^n = g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1}$. This way, the semigroup action of G may be understood either as a random walk inside $End^2(M)$ or as a (non-local) random perturbation inside $End^2(M)$. In both readings, the orbits correspond to projections in the fiber M of the orbits of the skew-product \mathcal{F}_G . More precisely, the shift (Σ_p^+, σ) allows us to identify each element $i_n \dots i_1$ of the free semigroup F_p and each $\underline{g} = g_{i_n} \dots g_{i_1} \in G$ satisfying $|\underline{g}| = n$ with f_{ω}^n for some suitable choice of $\omega \in \Sigma_p^+$: one must set $\omega_k = i_k$ for all $1 \leq k \leq n$. Observe, however, that for each concatenation $g_{i_n} \dots g_{i_1}$ we have several

choices of paths ω where we may fit the finite word i_1, \dots, i_n in the first n steps. These choices of ω define a cylinder in Σ_p^+ , we denote by $C(1; i_1, \dots, i_n)$, whose $\eta_{\underline{p}}$ measure is equal to $\frac{1}{p^n}$. We also notice that, although the F_p consists of finite concatenations of generators, the induced skew-product \mathcal{F}_G , with a dynamics driven by the shift on one-sided sequences endowed with the invariant probability measure $\eta_{\underline{p}}$, spreads with equal probability to typical orbits with respect to the random walk $\iota_*(\eta_{\underline{p}})$ on G .

Proposition 9.1. *Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 expanding map on a compact connected Riemannian manifold M , and let $S : G \times M \rightarrow M$ be the corresponding continuous semigroup action. Then there exists a Hölder-continuous nonnegative map $H_G : M \rightarrow \mathbb{R}$ such that $\nu_G = H_G \text{Leb}$ is an absolutely continuous stationary probability measure for the semigroup S and the symmetric random walk $R_{\underline{p}} = \iota_*(\eta_{\underline{p}})$.*

Proof. The linear Koopman and transfer operators associated to each random perturbation of f_ω^1 (which is C^2 expanding; see Remark 4.2) are defined by

$$U_\omega \psi = \psi(f_\omega^1) \quad \text{and} \quad \mathcal{L}_\omega(\psi)(x) = \sum_{f_\omega^1(y)=x} \frac{1}{|\det Df_\omega^1|} \psi(y)$$

for any Hölder-continuous map ψ and $x \in M$. Their iterates are, respectively,

$$U_\omega^n \psi = \psi(f_\omega^n) \quad \text{and} \quad \mathcal{L}_\omega^n(\psi)(x) = \sum_{f_\omega^n(y)=x} \frac{1}{|\det Df_\omega^n|} \psi(y).$$

Accordingly, the global transfer operators of the random perturbation of f_ω^1 are given by

$$\widehat{U}_G \psi = \int U_\omega \psi d\eta_{\underline{p}}(\omega) \quad \text{and} \quad \widehat{\mathcal{L}}_G(\psi)(x) = \int \mathcal{L}_\omega(\psi)(x) d\eta_{\underline{p}}(\omega)$$

for every Lebesgue-integrable observable $\psi : M \rightarrow \mathbb{R}$ and all $x \in M$. Taking into account that, as G_1 is finite, then the expanding estimates of the elements of G_1 are uniform, we may apply the classical results concerning random perturbations of expanding dynamics and deduce that:

Lemma 9.2. [36, Section 2] *There is a nonnegative Hölder function $H_G = \lim_{n \rightarrow +\infty} \widehat{\mathcal{L}}_G^n(\mathbf{1})$ which is a fixed point for the operator $\widehat{\mathcal{L}}_G$.*

Normalizing H_G so that $\int H_G d\text{Leb} = 1$, and setting $\nu_G = H_G \text{Leb}$, we obtain a probability measure in M absolutely continuous with respect to Lebesgue.

Lemma 9.3. ν_G is a stationary measure.

Proof. The measure ν_G satisfies $\int (\widehat{U}_G \psi) d\nu_G = \int \psi d\nu_G$ for every $\psi \in C^0(M)$, that is,

$$\int \left(\int (\psi \circ g_{\omega_1})(x) d\eta_{\underline{p}}(\omega) \right) d\nu_G(x) = \int \psi(x) d\nu_G(x).$$

Using Fubini-Tonelli theorem and taking into account that η_p is equally distributed, we may exchange the order of integration on the first term and obtain

$$\int \left(\int (\psi \circ \underline{g})(x) d\nu_G(x) \right) dR_{\underline{p}}(\underline{g}) = \int \psi(x) d\nu_G(x).$$

This confirms that ν_G is an $R_{\underline{p}}$ -stationary measure for the semigroup S (cf. (1)). \square

\square

In general one cannot expect the expanding maps of the semigroup to have common invariant probability measures; for that reason, the stationary probability measure is seldom invariant under all the elements of G . We also observe that, when studying the statistical stability of a dynamical system with respect to absolutely continuous invariant measures, one usually considers iterations of randomly chosen dynamics in a neighborhood of the original dynamics and randomness is given by a parameter assumed to belong to an interval and to be absolutely continuous with respect to the Lebesgue measure (see for instance [1] and references therein). In the previous result we did not require the C^2 -expanding maps to be close to each other and the randomness of the iterations is given by the fixed random walk.

We note that the notions of stationary measure for the semigroup action S and invariant measure for the skew-product \mathcal{F}_G are somehow linked, as the next result indicates.

Proposition 9.4. *Let η be a σ -invariant probability measure on Σ_p^+ and consider $R_\eta = \iota_*(\eta)$. Then ν is an R_η -stationary measure on M if and only if $\eta \times \nu$ is an \mathcal{F}_G -invariant probability measure.*

Proof. Given a continuous observable $\Psi : \Sigma_p^+ \times M \rightarrow \mathbb{R}$, we need to show that

$$\int (\Psi \circ \mathcal{F}_G) d(\eta \times \nu) = \int \Psi d(\eta \times \nu)$$

that is,

$$\int \int \Psi(\sigma(\omega), f_\omega^1(x)) d\eta(\omega) d\nu(x) = \int \int \Psi(\omega, x) d\eta(\omega) d\nu(x).$$

As ν is R_η -stationary, we have

$$\int \left(\int \Psi(\sigma(\omega), f_\omega^1(x)) d\eta(\omega) \right) d\nu(x) = \int \left(\int \Psi(\sigma(\omega), x) d\eta(\omega) \right) d\nu(x).$$

Moreover, if we consider, for a fixed $x \in M$, the map

$$\begin{aligned} V : \Sigma_p^+ &\rightarrow \mathbb{R} \\ \omega &\mapsto \int \Psi(\omega, x) d\nu(x) \end{aligned}$$

then, as η is σ -invariant and V is continuous, we obtain

$$\int V(\sigma(\omega)) d\eta(\omega) = \int V(\omega) d\eta(\omega)$$

that is,

$$\int \left(\int \Psi(\sigma(\omega), x) d\nu(x) \right) d\eta(\omega) = \int \left(\int \Psi(\omega, x) d\nu(x) \right) d\eta(\omega).$$

Therefore,

$$\int \left(\int \Psi(\sigma(\omega), f_\omega^1(x)) d\eta(\omega) \right) d\nu(x) = \int \int \Psi(\omega, x) d\nu(x) d\eta(\omega).$$

Conversely, if μ is a probability measure invariant under the skew-product such that $\mu = (\pi_{\Sigma_p^+})_*(\mu) \times (\pi_M)_*(\mu)$, where $\pi_{\Sigma_p^+}$ and π_M are the projections from $\Sigma_p^+ \times M$ onto the first and second coordinates, respectively, then clearly $(\pi_M)_*(\mu)$ is a $\iota_*((\pi_{\Sigma_p^+})_*(\mu))$ -stationary measure. \square

10. SELECTION OF MEASURES FOR SEMIGROUP ACTIONS

The action of a semigroup generated by more than one dynamics is not a dynamical system, thus it is not straightforward how to define equilibrium states and establish a variational principle that might relate topological and measure theoretical aspects of the semigroup action. Yet, under adequate hypothesis, a semigroup action can be embodied into a dynamical system whose topological and measure theoretical properties we may study and convey to the semigroup action.

From Section 8 recall that

$$h_{\text{top}}(S, \eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}) = P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{a}})$$

(cf. relations (28) and (38)). These two (different flavored) equalities justify the construction of maximal entropy measures for semigroup actions, arising from skew-product dynamics, that reflect the periodic data, the equidistribution among preimages or both.

Given an Hölder potential $\psi : M \rightarrow \mathbb{R}$, consider the fiberwise constant potential in the skew product \mathcal{F}_G defined as

$$\begin{aligned} \varphi = \varphi_\psi : \Sigma_p^+ \times M &\rightarrow \mathbb{R} \\ (\omega, x) &\mapsto \psi(x). \end{aligned}$$

Definition 10.1. We say that a probability measure ν on the Borel sets of M is an *equilibrium state for the semigroup action and ψ (arising from the skew-product dynamics)* if $\nu = (\pi_M)_*(\mu_\varphi)$, where μ_φ is the unique equilibrium state for the (topologically mixing Ruelle-expanding) map \mathcal{F}_G with respect to the fiberwise constant potential φ_ψ .

Remark 10.2. If $\nu = (\pi_M)_*(\mu_\varphi)$ as in the previous definition and $\mu_\varphi = (\mu_\omega)_{\omega \in \Sigma_p^+}$ is a disintegration along the measurable partition $(\pi^{-1}(\omega))_{\omega \in \Sigma_p^+}$, guaranteed by Rohlin's theorem then $\nu = \int_{\Sigma_p^+} \mu_\omega d\eta(\omega)$. Although this resembles an η -stationarity condition (recall ν is η -stationary if $\nu = \int_{\Sigma_p^+} (g_\omega)_* \nu d\eta(\omega)$) these two notions do not coincide necessarily. Indeed, if the semigroup G is generated by $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 -smooth expanding map, and $\psi \equiv 0$, then μ_φ is the measure of maximal entropy of \mathcal{F}_G and it is also its annealed equilibrium state with respect to ψ and $\eta_{\underline{a}}$, where

$\underline{a} = \left(\frac{\deg(g_1)}{\sum_{k=1}^p \deg(g_k)}, \frac{\deg(g_2)}{\sum_{k=1}^p \deg(g_k)}, \dots, \frac{\deg(g_p)}{\sum_{k=1}^p \deg(g_k)} \right)$. Moreover, the probability $(\pi_M)_*(\mu_\varphi)$ is the measure constructed in [9]. Observe also that, from this reference, one can infer that the maximal entropy measure on M is not, in general, a stationary measure.

The discussion in Subsection 8.1 suggests another approach. Given an Hölder continuous potential $\psi : M \rightarrow \mathbb{R}$ and the corresponding $\varphi = \varphi_\psi : \Sigma_p^+ \times M \rightarrow \mathbb{R}$, we may associate to any random walk on G (determined by a probability measure $\eta_{\underline{a}}$ on Σ_p^+) two probability measures on M which are the marginals on M of the unique annealed and quenched equilibrium states $\mu_{\underline{a},\varphi}^{(a)}$ and $\mu_{\underline{a},\varphi}^{(q)}$, which may be distinct [4, Section 2.4]. The projection $\pi_M : \Sigma_p^+ \times M \rightarrow M$ generates two measures on M , say $(\pi_M)_*(\mu_{\underline{a},\varphi}^{(a)})$ and $(\pi_M)_*(\mu_{\underline{a},\varphi}^{(q)})$, but observe that, even when the quenched and annealed states are different, it is not plain that they have different marginals on M . Since the physically observable measures are these M -marginals, it is worth exploring under what conditions on the semigroup action they coincide, in which case it would be natural to say that a probability measure ν on the Borel sets of M is an *equilibrium state for the semigroup action with respect to ψ and \underline{a}* if $\nu = (\pi_M)_*(\mu_{\underline{a},\varphi}^{(a)})$, where $\mu_{\underline{a},\varphi}^{(a)}$ is the unique annealed (or quenched) equilibrium state for the skew-product \mathcal{F}_G with respect to $\varphi = \varphi_\psi$.

For instance, it may happen that the annealed equilibrium state $\mu_{\underline{a},\varphi}^{(a)}$ for \mathcal{F}_G with respect to a potential φ and a non-trivial probability vector \underline{a} satisfies $\pi_{\mu_{\underline{a},\varphi}^{(a)}} = \eta_{\underline{a}}$, in which case $h^{\underline{a}}(\pi_{\mu_{\underline{a},\varphi}^{(a)}}) = 0$, and so $\mu_{\underline{a},\varphi}^{(a)}$ is also the quenched equilibrium state of φ with respect to \underline{a} (cf. [4, Proposition 2]). This occurs, for example, when G is generated by $G_1 = \{id, g_1, \dots, g_p\}$, where each g_i is a C^2 -smooth expanding map, and the potential φ is defined as $\varphi(\omega, x) = -\log |\det Dg_{\omega_1}|(x)$. In this particular case, the marginal on M of this common equilibrium has a disintegration in M which is almost everywhere absolutely continuous with respect to the Lebesgue (cf. [4, Remark 3.4]). Moreover, for the special vector \underline{p} , the operator

$$\mathbf{L}_{n,\varphi} = \frac{1}{p^n} \sum_{|\underline{g}|=n} \mathcal{L}_{\underline{g},\varphi} = \int_{\Sigma_p^+} \mathcal{L}_{\underline{g},\varphi} d\eta_{\underline{p}}(\omega)$$

coincides with the averaged Ruelle-Perron-Frobenius of [4]. However, this potential is not φ_ψ for any observable map ψ on M .

Preimage equidistributed measures. Consider a finite set $G_1 = \{id, g_1, \dots, g_p\}$, $p \geq 2$, of expanding maps in $End^2(M)$ and let G be the semigroup generated by G_1 . In analogy with the setting of a single Ruelle-expanding dynamical system (cf. Remark 4.2), one expects a maximal entropy measure for a semigroup action to be computed as a weak* limit of a special sequence of probability measures on M .

Example 10.3. It follows from [9] that, when η is the Bernoulli measure $\eta_{\underline{p}}$, the sequence of measures

$$\frac{1}{\lambda^n} \sum_{|\underline{g}|=n} \sum_{\underline{g}(y)=x} \delta_y$$

is weak* convergent to some probability measure (independently of x). Equivalently

$$\frac{1}{e^{\log \frac{\lambda}{p}}} \left[\frac{1}{p^n} \sum_{|\underline{g}|=n} \sum_{\underline{g}(y)=x} \delta_y \right], \quad (39)$$

where $\lambda = \deg(g_1) + \dots + \deg(g_d)$. This special case, which corresponds to a symmetric random walk η_p , should be compared with the convergence of the equidistributed measures of the form (2) for the skew-product \mathcal{F}_G .

As noticed before, for $\eta = \eta_{\underline{a}}$ and an Hölder potential $\underline{\varphi}$ on $\Sigma_p^+ \times M$, there is a natural stationary transfer operator $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ acting on $C^0(M)$ as in (7) and

$$sp(\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}) = sp(\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}) = \exp(P_{\text{top}}^{(a)}(\mathcal{F}_G, \varphi, \underline{a})).$$

Moreover, by [4, Proposition 3.1], the spectral features of the transfer operators $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ and $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}$ are strongly related. For instance, $\lambda_{\underline{a}, \underline{\varphi}} := \exp(P_{\text{top}}^{(a)}(\mathcal{F}_G, \varphi, \underline{a}))$ is the leading eigenvalue for the transfer operator $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^n$ acting on $C^r(M)$ ($r \geq 1$) with one-dimensional eigenspace generated by some $\rho \in C^r(M)$. The dual operator $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^*$ defined, for every continuous $\psi : M \rightarrow \mathbb{R}$ by

$$\int \psi d\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^* \eta = \int \tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}} \psi d\eta$$

has also a one-dimensional eigenspace associated to the leading eigenvalue $\lambda_{\underline{a}, \underline{\varphi}}$ generated by some probability measure γ on M . Moreover, $(\pi_M)_* \circ \hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^* = \tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^* \circ (\pi_M)_*$ and for each $x \in M$ the measures

$$\lambda_{\underline{a}, \underline{\varphi}}^{-n} (\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^n)_* \delta_x = \frac{1}{\lambda_{\underline{a}, \underline{\varphi}}^n} \int \sum_{g_{\omega_n} \dots g_{\omega_1}(y)=x} e^{S_n \varphi_{\underline{g}}(y)} \delta_y d\eta_{\underline{a}}(\omega_1, \dots, \omega_n) \quad (40)$$

obtained by averaging the preimages of x according to the random walk $\eta_{\underline{a}}$ are convergent to the measure $\rho(x) \cdot \gamma$ on M .

On the other hand, the annealed equilibrium state $\mu_{\varphi, \underline{a}}^{(a)}$ on $\Sigma_p^+ \times M$ is absolutely continuous with respect to a conformal measure for $\hat{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^*$: there exists a probability measure $\hat{\gamma}$ on $\Sigma_p^+ \times M$ so that $\tilde{\mathbf{L}}_{\underline{a}, \underline{\varphi}}^* \hat{\gamma} = \lambda_{\underline{a}, \underline{\varphi}} \hat{\gamma}$, that $\mu_{\varphi, \underline{a}}^{(a)} = \rho \hat{\gamma}$ and $(\pi_M)_* \hat{\gamma} = \gamma$ (cf. [4, Proposition 3.1(2) and Proposition 3.2]). Thus, it is natural to consider the marginal measure of $\mu_{\varphi, \underline{a}}^{(a)}$,

$$\nu_{\varphi, \underline{a}} := (\pi_M)_* \mu_{\varphi, \underline{a}}^{(a)} = \rho \gamma,$$

on M , which is a probability measure.

Given a random walk on G (determined by a σ -invariant probability measure on Σ_p^+), the previous information on the integrated and fibered transfer operators legitimates the following concept of maximal entropy measure for the semigroup action with respect to the fixed random walk. Another approach may be found in [15], although not expressing the variational connections in terms of invariant measures; see [3] for details.

Notice that the major advantage of this definition is that (40) defines $\nu_{\varphi, \underline{a}}$ intrinsically, using the generators of the semigroup. Using (30), the next proposition summarizes the main link maximal entropy measures for the semigroup S and for the skew-product.

Proposition 10.4. *Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$ of C^2 expanding maps on a compact connected Riemannian manifold M . Consider the continuous semigroup action $S : G \times M \rightarrow M$ and the random walk $R_{\underline{a}} = \iota_*(\eta_{\underline{a}})$. Then there is a probability measure $\nu_{0, \underline{a}}$ of maximal entropy for the semigroup S with respect to the random walk $R_{\underline{a}}$ (arising from the skew product dynamics) and*

$$\nu_{0, \underline{a}} = (\pi_M)_*(\mu_{\underline{a}}^{(a)}).$$

Moreover, if $\underline{a} = \underline{m}$, then $\mu_{\underline{m}}^{(q)} = \mu_{\underline{m}} = \mu_{\underline{p}}^{(a)}$ and the following equalities hold:

$$\nu_{0, \underline{a}} = (\pi_M)_*(\mu_{\underline{m}}) = (\pi_M)_*(\mu_{\underline{p}}^{(a)}) = (\pi_M)_*(\mu_{\underline{m}}^{(q)}).$$

This ends the proof of Theorem C. One should mention that some similar expressions can be written in the case of locally constant potentials. Nevertheless we shall not need or use this fact here.

Questions. Is there a generalized version of the previous proposition for other potentials φ ? What if $\eta = \eta_{\underline{a}}$ with $\underline{a} \neq \underline{p}$? Recall also that ν is an η -stationary measure if $\nu = \int_{\Sigma_p^+} (g_\omega)_* \nu d\eta(\omega)$, a condition that resembles (40): how do these properties relate? Are there equilibrium states that do not arise from skew-product dynamics? Do all equilibrium measures have some Gibbs property?

Periodic masses convergence. The previous Proposition 10.4 establishes the existence of a maximal entropy measure for the semigroup action S that arises from skew-product dynamics and can be intrinsically defined by a process of averaging preimages (cf. (40)). It is a relevant issue to understand if such maximal entropy measures can also arise from the periodic points for the finite time sequential dynamics.

As mentioned before, the skew-product \mathcal{F}_G is a Ruelle expanding map, $h_{\text{top}}(S, \eta_{\underline{a}}) = P_{\text{top}}(\mathcal{F}_G, \varphi_{\underline{a}})$ and $\mu_{\underline{m}} = \mu_{\underline{p}}^{(a)}$. Consequently, we obtain the following:

Proposition 10.5. *Let G be the semigroup generated by a set $G_1 = \{id, g_1, \dots, g_p\}$ of C^2 expanding maps on a compact connected Riemannian manifold M . Consider the continuous semigroup action $S : G \times M \rightarrow M$ and the symmetric random walk $R_{\underline{p}} = \iota_*(\eta_{\underline{p}})$. Then there is a probability measure $\nu_{\varphi, \underline{p}}$ of maximal entropy for the semigroup S with respect to the random walk $R_{\underline{p}}$ (arising from the skew product dynamics) and*

$$\nu_{0, \underline{p}} = \lim_{n \rightarrow \infty} e^{-n h_{\text{top}}(S, \eta_{\underline{p}})} \sum_{\sigma^n(\omega) = \omega} \sum_{g_{\omega_n} \dots g_{\omega_1}(x) = x} \delta_x.$$

In particular,

$$\nu_{0, \underline{p}}(A) = \lim_{n \rightarrow \infty} e^{-n h_{\text{top}}(S, \eta_{\underline{p}})} \sum_{\sigma^n(\omega) = \omega} \#\{Fix(g_{\omega_n} \dots g_{\omega_1}) \cap A\}$$

for any measurable set $A \subset M$ satisfying $\nu_{0,p}(\partial A) = 0$.

Proof. In this setting, $\mu_p^{(a)} = \mu_{\underline{m}}$. By the continuity of the push-forward map $(\pi_M)_*$ and the asymptotic growth rate of periodic orbits, we conclude that

$$\begin{aligned} \nu_{0,p} &= (\pi_M)_*(\mu_p^{(a)}) = (\pi_M)_*(\mu_{\underline{m}}) = (\pi_M)_*\left(\lim_{n \rightarrow \infty} \frac{\sum_{(\omega,x) \in \text{Fix}(\mathcal{F}_G^n)} \delta_{(\omega,x)}}{\#\text{Fix}(\mathcal{F}_G^n)}\right) \\ &= \lim_{n \rightarrow \infty} (\pi_M)_*\left(\frac{\sum_{\sigma^n(\omega)=\omega} \sum_{g_{\omega_n} \dots g_{\omega_1}(x)=x} \delta_{(\omega,x)}}{\sum_{\sigma^n(\omega)=\omega} \#\text{Fix}(g_{\omega_n} \dots g_{\omega_1})}\right) \\ &= \lim_{n \rightarrow \infty} e^{-nh_{\text{top}}(S, \eta_p)} \sum_{\sigma^n(\omega)=\omega} \sum_{g_{\omega_n} \dots g_{\omega_1}(x)=x} \delta_x. \end{aligned}$$

The second assertion is immediate from the definition of the weak* convergence. \square

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